Surface node method for the peridynamic simulation of elastodynamic problems with Neumann boundary conditions

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Abstract. Peridynamics is a nonlocal theory that can effectively handle discontinuities, including crack initiation and propagation. However, near the boundaries, the incomplete nonlocal regions are the cause of the peridynamic surface effect, resulting in unphysical stiffness variation. Additionally, imposing local boundary conditions in a peridynamic (nonlocal) model is often necessary. To address these issues, the surface node method has been proposed for improving accuracy near the boundaries of the body. Although this method has been verified for a variety of problems, it has not been applied for elastodynamic problems involving Neumann boundary conditions. In this work we show a numerical example of this case, comparing the results with the corresponding peridynamic analytical solution. The numerical results exhibit no stiffness variations near the boundaries throughout the entire simulation timespan. Therefore, we conclude that the surface node method allows to effectively solve elastodynamic peridynamic problems involving Neumann boundary conditions, with improved accuracy near the boundaries.

Introduction to peridynamics

Peridynamics (PD) is a nonlocal continuum theory based on integro-differential equations in which discontinuities, such as cracks, in the displacement field can arise and evolve without mathematical inconsistencies [1,2]. In a PD body B, two points interact through a so-called *bond* if their distance is smaller than δ, named *horizon size*. The PD equation of motion for a generic point x at a time instant t in a 1D, homogeneous, linear elastic body [3] is given as

$$\ddot{u}(x,t) = v^2 \int_{H_x} \frac{u(x',t)-u(x,t)}{\delta(x'-x)^2} \mathrm{d}x', \tag{1}$$

where $H_x = \{x' \in B : |x' - x| \leq \delta\}$ is the set of points x' interacting with point x, u is the displacement, $\ddot{u}$ is the acceleration, and v is the wave speed.

By using the meshfree method with a uniform grid spacing Δx [4], in which every node represents a cell of length Δx (see Fig. 1), Eq. 1 is discretized in space as

$$\ddot{u}(x_i,t) = \frac{v^2}{\delta} \sum_{j \in H_i} \frac{u(x_j,t)-u(x_i,t)}{(x_j-x_i)^2} \beta_{ij} \Delta x \,, \tag{2}$$

where x_i and x_j are respectively the coordinates of node i and any node j within the neighborhood H_i of node i, and β_{ij} is the quadrature coefficient, namely the fraction of cell of node j which lies within the neighborhood H_i [5]. The explicit central difference method is used for time integration [4]:

$$u(x_i, t_{n+1}) = 2u(x_i, t_n) - u(x_i, t_{n-1}) + \frac{(v\Delta t)^2}{\delta} \sum_{j\in H_i} \frac{u(x_j,t)-u(x_i,t)}{(x_j-x_i)^2} \beta_{ij}\Delta x \,, \tag{3}$$

where Δt is the time step size and n stands for the index of the current time step.

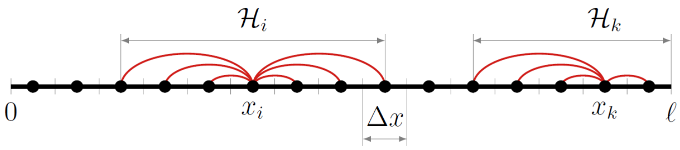

Figure 1: Each node (black dots) in the peridynamic body represents a cell of length Δx and interacts with all the nodes within its neighborhood through bonds (red lines).

As shown in Fig. 1, the nodes near the boundary of the body have an incomplete neighborhood. Due to this fact, the stiffness properties of the nodes close to the boundary are different from those of the nodes in the bulk. This undesired phenomenon is called *PD surface effect* [6-9]. Furthermore, boundary conditions in PD models should be imposed over a finite-thickness layer, in contrast with experimental measurements which are performed only at the boundary. Therefore, we use the Surface Node Method (SNM) to solve these issues [7-9].

Overview of the Surface Node Method

As shown in Fig. 2, the *fictitious nodes* are added in the peridynamic model to complete the neighborhoods of the interior nodes near the boundaries of the body. Furthermore, the *surface nodes* are introduced at the boundaries of the body to impose the boundary conditions.

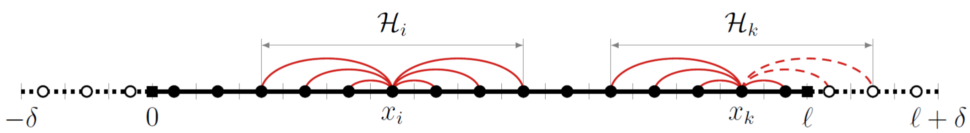

Figure 2: The fictitious nodes (empty dots) are introduced to reduce the PD surface effect by completing the neighborhoods of the interior nodes (solid dots). The surface nodes (solid squares) are introduced to impose the peridynamic boundary conditions through the fictitious bonds (red dashed lines).

The displacements of the fictitious nodes are determined by extrapolation from the displacements of the interior nodes. By using, for example, the linear Taylor-based method [7-9], the displacement of any fictitious node f is given as

$$u(x_f, t_n) = u(x_s, t_n) + (x_f - x_s) \frac{u(x_s,t_n)-u(x_p,t_n)}{x_s-x_p} \,, \tag{4}$$

where s is the index of the closest surface node and p is the index of the interior node closest to that surface node.

The surface nodes do not have interactions (bonds) with other nodes, the state of the fictitious bonds crossing them is governed by the equation of the force flux [7-9]:

$$\tau(x_s, t_n) = \frac{v^2}{\delta} \sum_{ij\in \mathrm{I}} \frac{u(x_j,t_n)-u(x_i,t_n)}{(x_j-x_i)^2} \beta_{ij}\Delta x^2, \tag{5}$$

where $\tau(x_s, t_n)$ is the force flux at the surface node s and I is the set of all the fictitious bonds with positive direction crossing the boundary at x_s. Thanks to Eq. 5, constraints and loads can be applied to the surface nodes as one would do in a local model.

Numerical example

For the first time, we present a numerical example of a peridynamic model making use of the SNM to solve an elastodynamic problem involving Neumann boundary conditions. The initial boundary value problem is given as

$$\begin{cases} \ddot{u}(x,t) = \frac{v^2}{\delta}\int_{H_x} \frac{u(x',t)-u(x,t)}{(x'-x)^2}\mathrm{d}x' & \text{for } 0 < x < \ell,\ t > 0\,, \\ u(0,t) = 0,\ \tau(\ell,t) = 0 & \text{for } t > 0\,, \\ u(x,0) = 0.02e^{-100\left(\frac{x-0.5}{\ell}\right)^2},\ \dot{u}(x,0) = 0 & \text{for } 0 < x < \ell\,, \end{cases} \tag{6}$$

where $\dot{u}$ is the velocity and ℓ is the length of the peridynamic body. The analytical solution is computed by means of the method of separation of variables [3]:

$$u(x,t) = \sum_{m=1,3,5,\dots}^{\infty} \frac{0.004\sqrt{\pi}}{\ell}\sin\left(\frac{k_m\ell}{2}\right)e^{\frac{-k_m{}^2}{400}}\sin(k_m x)\cos\left(vt\sqrt{\frac{2}{\delta^2}[k_m\delta\ \mathrm{Si}(k_m\delta)+\cos(k_m\delta)-1]}\right), \tag{7}$$

where $k_m = \frac{m\pi}{2\ell}$ and $\mathrm{Si}(\cdot)$ is the sine integral function. The analytical solution in Eq. 7, truncated at $m = 80$, will be used as a comparison for the numerical results in Fig. 3.

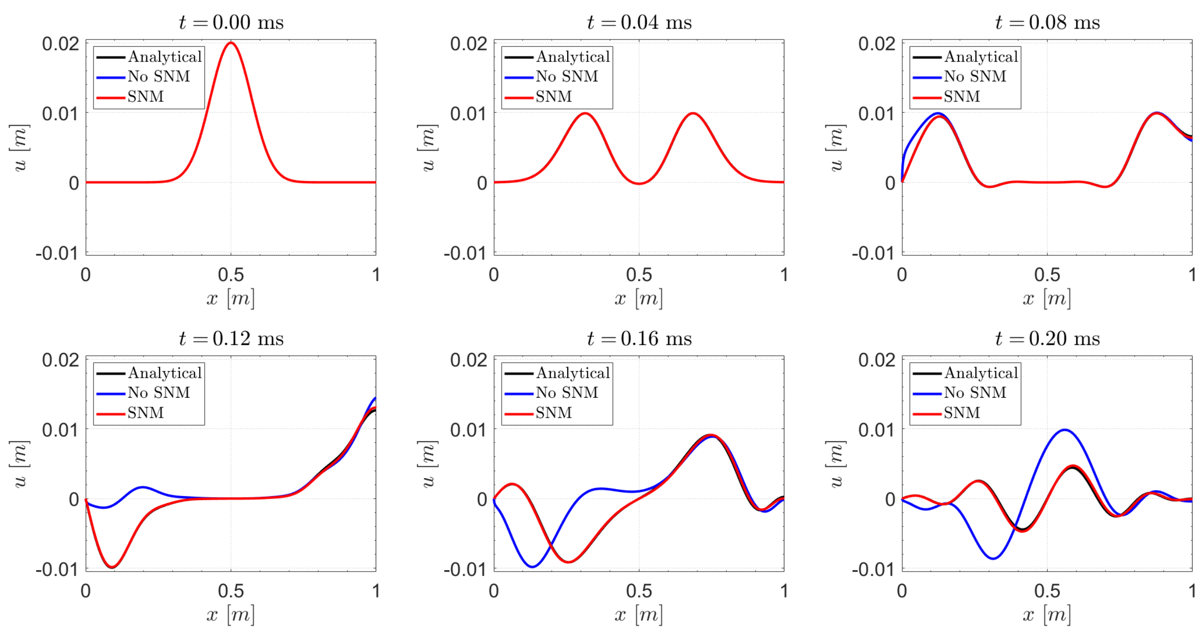

Figure 3: Plots of the propagating wave at different instants of time t for the analytical solution and the peridynamic models with and without the use of the Surface Node Method (SNM).

Fig. 3 shows the numerical results of the peridynamic model with and without the use of the SNM. If the SNM is not employed, the loads and the constraints are applied directly to the interior nodes closest to the boundary. The model parameters used to obtain those results are the following: $v = 5000$ m/s, $\ell = 1$ m, $\delta = 0.1$ m, $\Delta x = 0.001$ m, and $\Delta t = 0.2$ μs. As shown in Fig. 3, it is evident that the model with the SNM provides results closer to the analytical peridynamic solution. The major differences are noticeable in the region near the end of the bar, where Dirichlet boundary conditions are imposed. However, without employing any boundary correction, non-negligible errors are also present at the other end of the bar, where Neumann boundary conditions are applied. Thus, the SNM allows to considerably reduce the numerical errors due to the PD surface effect and the imposition of the boundary conditions.

Conclusions

In this work, we numerically solved a 1D peridynamic elastodynamic problem involving Neumann boundary conditions by using the Surface Node Method (SNM) to mitigate the PD surface effect and impose the boundary conditions. The analytical solution to this problem has been derived thanks to the method of separation of variables. The numerical results show that the use of the SNM significantly reduces the errors near the boundaries of the body when compared to the corresponding model without boundary corrections, both where Dirichlet and Neumann boundary conditions are applied.

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