Exact solutions for free vibration analysis of train body by Carrera unified formulation (CUF) and dynamic stiffness method (DSM)

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Abstract. A novel approach for free vibration analysis of train body structures is introduced by using the Carrera Unified Formulation (CUF) and Dynamic Stiffness Method (DSM). Higher-order kinematic fields are developed using the Carrera Unified Formulation, which allows for straightforward implementation of any-order theory without the need for ad hoc formulations, in the case of beam theories. In particular, the parallel axis theorem is introduced on the basis of the Taylor expansion cross-sectional displacement variables, which unifies the different shape subsections of the train into the same coordinate system. The Principle of Virtual Displacements is used to derive the governing differential equations and the associated natural boundary conditions. An exact dynamic stiffness matrix is then developed by relating the amplitudes of harmonically varying loads to those of the responses. Finally, the Wittrick–Williams (WW) algorithm was used to carry out the free vibration analysis of the train body and the natural frequencies and corresponding modal shapes are presented.

Introduction

Train body frame is an important part of the train as a load-carrying system. The operating environment of high-speed trains is complex, with the increase of speed, the vibration of train body becomes more and more obvious, which has a great impact on the stability, comfort and safety of train operation. As the first step in the optimal design of train body structures, free vibration analysis is an important part of the analysis of train dynamic characteristics. At present, high-speed trains adopt the concept of modular design, and analyze the structure of different parts of the frame separately. The dynamic analysis of the train body frame is still in its infancy [1], so it is necessary to establish an efficient and accurate modeling method and analysis method for the body frame.

As a typical frame structure in engineering, the shape of the train body structure is extremely complex. In order to be able to calculate, certain assumptions and simplifications are often adopted [2]. Among many approximate analysis methods, the finite element method (FEM) is undoubtedly the most extensive and effective numerical calculation method. The finite element (FE) modeling method of the train body can choose refined modeling and equivalent modeling. Refined modeling is to appropriately simplify the train body structure under the premise of retaining the main shape, and reduce the number of units and calculation time of the model while reflecting the actual structural characteristics of the train body as much as possible. Sung-Cheol Yoon [3], H.Kurtaran [4], Wang Wei [5] and others studied the refined modeling of train body structure respectively,

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and on the basis of the model, the analysis and calculation of strength and mode were carried out. The equivalent modeling is to perform equivalent processing on the structure of the train body. There is a certain difference between the shape and the actual model. The mechanical parameters are derived and calculated according to the actual structural materials. The equivalent model has advantages in terms of the number of units compared to the refined modelling. D.Ribeiro[1] and others established the equivalent single-layer plate model of the BBN train body in the Alpha train. Shen Zhenhong [6] used different methods to establish the train sandwich plate structure. The finite element model of different modeling methods is compared with the deviation of the calculation results of the solid element method, and the feasibility of the lamination theory modeling method is verified. However, the FEM essentially obtains approximate solutions by dividing units. The above two methods still require a certain number of units to meet the accuracy requirements for complex frame structures, and their high calculation costs are hard to accept for the optimal design of train body structures.

The present work is intended to provide a more powerful approach for the free vibration analysis of train body as a beam structure through the application of the Carrera Unified Formulation (CUF) and dynamic stiffness method (DSM) in a much broader context by allowing for the cross-sectional deformation. CUF is a hierarchical formulation that considers the order of ofthe model, *N*, as a free-parameter (i.e. as an input) of the analysis or in other words, refined models are obtained without having the need for any ad hoc formulation [7-9]. On this basis of the Taylor expansion (TE), we introduce the parallel axis theorem (PAT) for train body structure, which broadens the applicable field of TE. On the other hand, the DSM is appealing in dynamic analysis because unlike the FEM, it provides exact solution of the equations of motion of a structure once the initial assumptions on the displacements field have been made. This essentially means that, unlike the FEM and other approximate methods, the model accuracy is not unduly compromised when a small number of elements are used in the analysis.

In this work, 1D higher-order Dynamic Stiffness (DS) elements based on CUF are extended and applied to the free vibration analysis of train body. In the next section, CUF and PAT is introduced and higher-order models are formulated. The principle of virtual displacements is then used to derive the equations of motion and the natural boundary conditions, which are subsequently expressed in the frequency domain by assuming a harmonic solution. After the resulting system of ordinary differential equations of second order with constant coefficients is solved, the frequency dependent DS matrix of the system is derived. Finally, the algorithm of Wittrick and Williams is applied to extrapolate the free vibration characteristics of train body.

1D unified formulation

Preliminaries

Within the framework of the CUF, the displacement field $\boldsymbol{u}(x, y, z; t)$ can be expressed as

$$\boldsymbol{u}(x, y, z; t) = F_\tau(x, z)\boldsymbol{u}_\tau(y; t), \quad \tau = 1,2, \dots, M. \tag{1}$$

where F_τ are the functions of the coordinates *x* and *z* on the cross-section. $\boldsymbol{u}_\tau$ is the vector of the generalized displacements, *M* stands for the number of the terms used in the expansion, and the repeated subscript, τ, indicates summation. The choice of F_τ determines the class of the 1D CUF model that is required and subsequently to be adopted. According to Eq. 1, TE (Taylor expansion) 1D CUF models consist of a MacLaurin series that uses the 2D polynomials $x^i z^j$ as F_τ functions, where *i* and *j* are positive integers. For instance, the displacement field of the second-order (N = 2) TE model can be expressed as

$$u_x = u_{x_1} + xu_{x_2} + zu_{x_3} + x^2u_{x_4} + xzu_{x_5} + z^2u_{x_6},$$
$$u_y = u_{y_1} + xu_{y_2} + zu_{y_3} + x^2u_{y_4} + xzu_{y_5} + z^2u_{y_6}, \tag{2}$$
$$u_z = u_{z_1} + xu_{z_2} + zu_{z_3} + x^2u_{z_4} + xzu_{z_5} + z^2u_{z_6}.$$

The order N of the expansion is set as an input option of the analysis; the integer N is arbitrary and it defines the order the beam theory.

Governing equations of the N-order TE model and parallel axis theorem
The principle of virtual displacements is used to derive the equations of motion.

$$\delta L_{\mathrm{int}} = \int_V \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma} \mathrm{d}V = -\delta L_{\mathrm{ine}}\,. \tag{3}$$

where $\boldsymbol{\sigma}$ is stress, $\boldsymbol{\epsilon}$ is strain, L_{int} stands for the strain energy and L_{ine} is the work done by the inertial ladings. δ stands for the usual virtual variation operator. After integrations by part, Eq. 3 becomes

$$\delta L_{\mathrm{int}} = \int_L \delta\mathbf{u}_\tau^T \mathbf{K}^{\tau s}\mathbf{u}_s \mathrm{d}y + [\delta\mathbf{u}_\tau^T \mathbf{\Pi}^{\tau s}\mathbf{u}_s]_{y=0}^{y=L}. \tag{4}$$

where $\mathbf{K}^{\tau s}$ is the differential linear stiffness matrix and $\mathbf{\Pi}^{\tau s}$ is the matrix of the natural boundary conditions in the form of 3×3 fundamental nuclei. Due to space reasons, the $\mathbf{K}^{\tau s}$ matrix and $\mathbf{\Pi}^{\tau s}$ matrix are not expanded in detail which can be referred to [9]. However, the critical part of these matrices is the solution of the cross-sectional moment parameter $E^{\alpha\beta}_{\tau,\theta S,\zeta}$.

$$E^{\alpha\beta}_{\tau,\theta S,\zeta} = \int_\Omega \tilde{C}_{\alpha\beta} F_{\tau,\theta} F_{S,\zeta} \mathrm{d}\Omega. \tag{5}$$

where $\tilde{C}_{\alpha\beta}$ is the coefficient matrix related to the Young modulus, the Poisson ratio, and fiber orientation angle. For the integration of cross-section functions, in general, the Taylor expansion is only applicable to the whole cross-section. However, as shown in the Fig.1, the cross-section of the train body contains both the Cartesian coordinate system and the polar coordinate system, which need to be integrated separately. Therefore, we introduce the concept of the parallel axis theorem and the specific steps are as follows

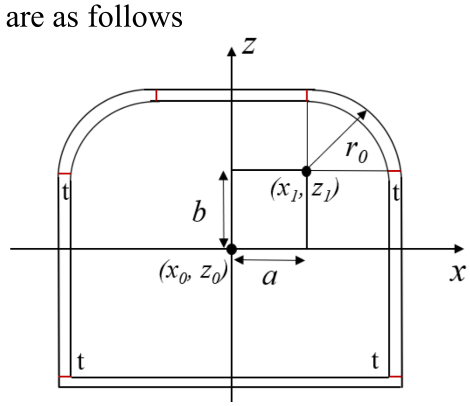


Fig.1 Cross-section of the train body

For the Taylor expanded body section integral, when the midpoint of the integral is (x_0, z_0), it can be expressed as

$$\int F_\tau F_S d\Omega = \sum_{m=0}^{N} \sum_{c=0}^{N} \int x^m z^n\, x^c z^d d\Omega. \tag{6}$$

where n=N-m, d=N-c. Coordinates (x_0, z_0) have the following relationship with (x_1, z_1):

$$x_1 - x_0 = a, z_1 - z_0 = b. \tag{7}$$

Thus, to represent the circular area in the polar coordinates, each term of the summation in Eq. 6 can be rewritten as

$$\begin{aligned}&\int_\Omega (x+a)^m (z+b)^n (x+a)^c (z+b)^d d\Omega \\ &= \int \sum_{i=0}^{M} C_m^i x^i a^{m-i} \sum_{j=0}^{n} C_n^j z^j b^{n-j} \sum_{p=0}^{t} C_c^p x^p a^{c-p} \sum_{q=0}^{d} C_d^q z^q b^{d-q} d\Omega \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{p=0}^{c} \sum_{q=0}^{d} C_m^i C_n^j C_c^p C_d^q a^{m-i} b^{n-j} a^{c-p} b^{d-q} \int x^i z^j x^p z^q d\Omega.\end{aligned} \tag{8}$$

where C is the binomial coefficient and the integral part in Eq. 8 can be further converted to solve in polar coordinates

$$\begin{aligned}&\int x^i z^j x^p z^q d\Omega = \int r^{i+j+p+q} \cos^{i+p}\ \theta \sin^{j+q}\ \theta d\Omega \\ &= \int_{r_1}^{r_0} \int_{\theta_1}^{\theta_2} r^{i+j+p+q} \cos^{i+p}\theta \sin^{j+q}\theta d\theta dr\,.\end{aligned} \tag{9}$$

where r_0 is the radius of the outer side of the ring. Based on the above method, the cross-sectional moment parameter $E_{\tau,\theta s,\zeta}^{\alpha\beta}$ of the train body section can be derived.

Then, the virtual variation of the inertial work is given by

$$\delta L_{\text{ine}} = \int_L \delta \mathbf{u}_\tau \int_\Omega \rho F_\tau F_S \mathrm{d}\Omega \ddot{\mathbf{u}}_s \mathrm{d}y = \int_L \delta \mathbf{u}_\tau \mathbf{M}^{\tau s} \ddot{\mathbf{u}}_s \mathrm{d}y. \tag{10}$$

where $\mathbf{M}^{\tau s}$ is the differential linear mass matrix. And the explicit form of the governing equations is

$$\begin{aligned}\delta u_{x\tau}&: -E_{\tau s}^{66} u_{xs,yy} + \left(E_{\tau,xs}^{26} - E_{\tau s_{,x}}^{26}\right) u_{xs,y} + \left(E_{\tau_{,x}s_{,x}}^{22} + E_{\tau_{,z}s_{,z}}^{44}\right) u_{xs} \\ &\quad -E_{\tau s}^{36} u_{ys,yy} + \left(E_{\tau,x}^{23} - E_{\tau s_{,x}}^{66}\right) u_{ys,y} + \left(E_{\tau,xs_{,x}}^{26} + E_{\tau_{,z}s_{,z}}^{45}\right) u_{ys} \\ &\quad +\left(E_{\tau_{z,z}s}^{45} - E_{\tau s_{,z}}^{16}\right) u_{zs,y} + \left(E_{\tau_{,z}s_{,x}}^{44} + E_{\tau_{,x}s_{,z}}^{12}\right) u_{zs} = -E_{\tau s}^{\rho} \ddot{u}_{xs}, \\ \delta u_{y\tau}&: -E_{\tau s}^{36} u_{xs,yy} + \left(E_{\tau,xs}^{66} - E_{\tau s,x}^{23}\right) u_{xs,y} + \left(E_{\tau,x}^{26} s_{,x} + E_{\tau,z}^{45} s_{,z}\right) u_{xs} \\ &\quad -E_{\tau s}^{33} u_{ys,yy} + \left(E_{\tau,x}^{36} - E_{\tau s_{,x}}^{36}\right) u_{ys,y} + \left(E_{\tau,xs_{,x}}^{66} + E_{\tau_{,z}s_{,z}}^{55}\right) u_{ys} \\ &\quad +\left(E_{\tau,z}^{55} - E_{\tau s,z}^{13}\right) u_{zs,y} + \left(E_{\tau,xs,z}^{16} + E_{\tau,z}^{45}\right) = u_{zs} = -E_{\tau s}^{\rho} \ddot{u}_{ys}, \\ \delta u_{z\tau}&: \left(E_{\tau,z}^{16} - E_{\tau s_{,z}}^{45}\right) u_{xs,y} + \left(E_{\tau,xs,z}^{44} + E_{\tau,z}^{12} s_{,x}\right) u_{xs} \\ &\quad +\left(E_{\tau,z}^{13} - E_{\tau s_{,z}}^{55}\right) u_{ys,y} + \left(E_{\tau_{,x}s_{,z}}^{45} + E_{\tau,z}^{16} s_{,x}\right) u_{ys} - E_{\tau s}^{55} u_{zs,yy} \\ &\quad +\left(E_{\tau,xs}^{45} - E_{\tau s,x}^{45}\right) u_{zs,y} + \left(E_{\tau_{,x}s,x}^{44} + E_{\tau,zs_{,z}}^{11}\right) u_{zS} = -E_{\tau s}^{\rho} \ddot{u}_{zs}.\end{aligned} \tag{11}$$

where

$$E_{\tau s}^{\rho} = \int_\Omega \rho F_\tau F_S \mathrm{d}\Omega. \tag{12}$$

Double over dots stand for the second derivative with respect to time (t). Letting $\mathbf{P}_\tau = \{P_{x\tau}\ \ P_{y\tau}\ \ P_{z\tau}\}^T$ to be the vector of the generalized forces, the natural boundary conditions are

$$\begin{aligned}
&\delta u_{x\tau}: P_{xs} = E_{\tau s}^{66} u_{xs,y} + E_{\tau s,x}^{26} u_{xs} + E_{\tau s}^{36} u_{ys,y} + E_{\tau s,x}^{66} u_{ys} + E_{\tau s,z}^{16} u_{zs},\\
&\delta u_{y\tau}: P_{ys} = E_{\tau s}^{36} u_{xs,y} + E_{\tau s,x}^{23} u_{xs} + E_{\tau s}^{33} u_{ys,y} + E_{\tau s,x}^{36} u_{ys} + E_{\tau s,z}^{13} u_{zs},\\
&\delta u_{z\tau}: P_{zs} = E_{\tau s,z}^{45} u_{xs} + E_{\tau s,z}^{55} u_{ys} + E_{\tau s}^{55} u_{zs,y} + E_{\tau s,x}^{45} u_{zs}.
\end{aligned} \tag{13}$$

For a fixed approximation order *N*, Eq. 11 and 13 have to be expanded using the indices τ and s in order to obtain the governing differential equations and the natural boundary conditions of the desired model.

In the case of harmonic motion, the solution of Eq. 11 is sought in the form

$$\mathbf{u}_s(y;t) = \mathbf{U}_s(y) e^{\mathrm{i}\omega t}. \tag{14}$$

where $\mathbf{U}_s(y)$ is the amplitude function of the motion, ω is an arbitrary circular or angular frequency, and i is $\sqrt{-1}$. The formulation of the equilibrium equations and the natural boundary conditions in the frequency domain can be obtained by substituting Eq. 14 into Eq. 11.

Dynamic stiffness formulation

In Section 2, the ordinary differential equations of the beam in free vibration have been derived and the procedure to obtain the Dynamic Stiffness (DS) matrix for a structural problem can be summarized as follows: (i) Seek a closed form analytical solution of the governing differential equations of the structural element; (ii) Apply a number of general boundary conditions equal to twice the number of integration constants in algebraic form, which are usually the nodal displacements and forces; (iii) Eliminate the integration constants by relating the amplitudes of the generalized nodal forces to the corresponding generalized displacements generating the DS matrix $\boldsymbol{\mathcal{K}}$. For the sake of brevity, the expressions for the DS matrix $\boldsymbol{\mathcal{K}}$ are not reported here, but can be found in standard texts, see for example Pagani [9]. It should be noted that the DS matrix consists of both the inertia and stiffness properties of the structure element unlike the FEM for which they are separately identified.

The DS matrix $\boldsymbol{\mathcal{K}}$ is the basic building block to compute the exact natural frequencies of a higher-order beam. The DSM has also many of the general features of the FEM. In particular, it is possible to assemble elemental DS matrices to form the overall DS matrix of any complex structure consisting of beam elements. The global DS matrix can be written as

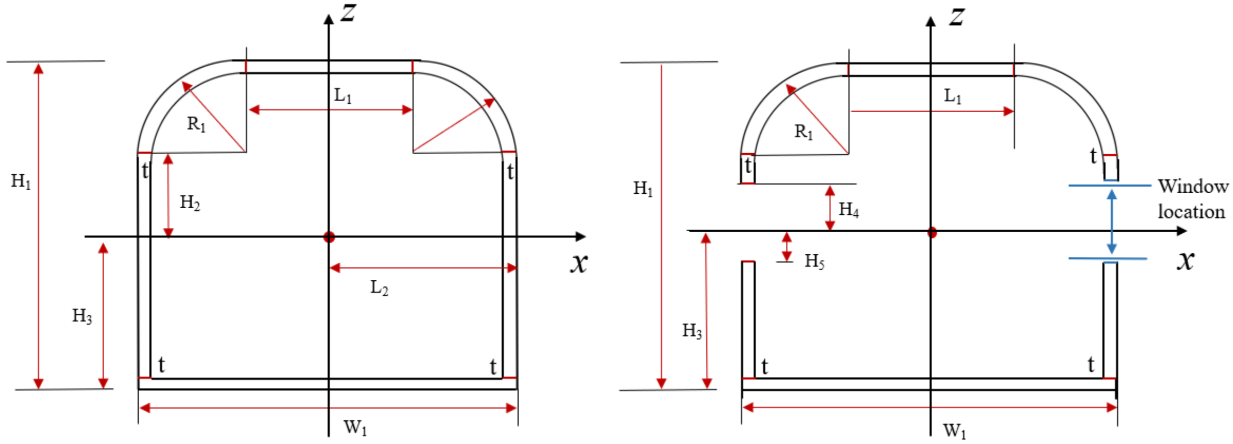
$$\bar{\boldsymbol{P}}_G = \mathcal{K}_G \bar{\boldsymbol{U}}_G. \tag{15}$$

where $\mathcal{K}_G$ is the square global DS matrix of the final structure. For the sake of simplicity, the subscript "*G*" is omitted hereafter. The train body structure can be regarded as beams of different cross-sectional forms, and the whole train body structure can be obtained by simply assembling it like FEM. The boundary conditions can be applied by using the well-known penalty method (often used in FEM) or by simply removing rows and columns of the stiffness matrix corresponding to the degrees of freedom which are zeroes. Due to the presence of higher-order degrees of freedom at each interface, a multitude of boundary conditions can be applied at the required nodes.

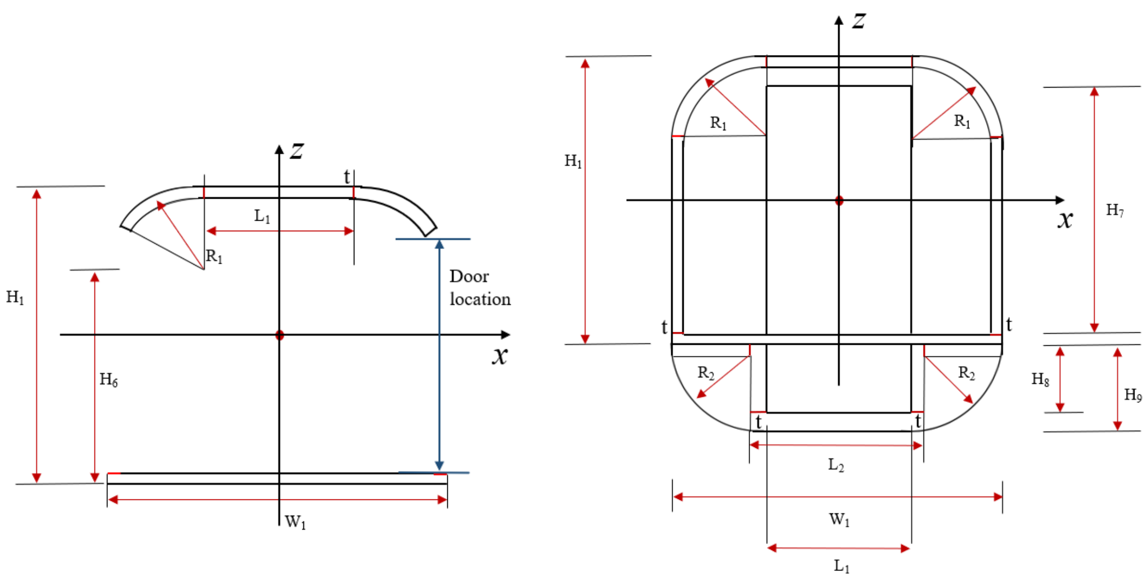
The Wittrick–Williams algorithm is used to solve the transcendental (nonlinear) eigenvalue problem generated by the DSM. Once the natural frequencies are calculated and the associated global DS matrix is obtained, the complete displacement field can be generated as a function of *x, y, z* and the time *t*. Clearly, the plot of the required mode and required element can be visualized on a fictitious 3D mesh. By following this procedure it is possible to compute the exact mode shapes using just one element which is impossible in FEM.

Numerical Results

A train body with four types cross-sections such as the one shown in Fig. 2 is considered. The four types represent the cross-section of the body frame, window, door and end wall respectively. The material data are the Young modulus, $E = 75\text{GPa}$, the Poisson ratio, $\nu = 0.33$, material density, $\rho = 2700\text{kgm}^{-3}$. The cross-sectional data are L_1=1.6m, L_2=1.65m, L_3=1m, L_4=2m, W_1=3.3m, H_1=2.55m, H_2=0.425m, H_3=1.275m, H_4=0.4m, H_5=0.2m, H_6=1.7m, H_7=2m, H_8=0.8m, H_9=1.04m, R_1=0.85m, R_2=0.75m, t=0.02m. Distribution of cross-section types in the y-direction (lengthwise) is shown in Fig. 3. The bodies of 4 types cross-sections are combined into a complete train structure which the length is 18m.



(a) Cross section of body frame (Type 1)   (b) Cross section of body frame with windows (Type 2)



(c) Cross section of body frame with doors (Type 3)   (d) Cross section of body frame with end wall (Type 4)

Fig. 2 Cross section of four typical train body frames

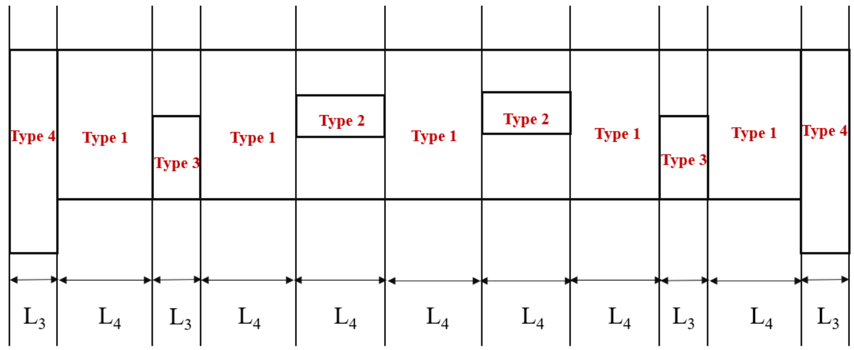


Fig.3 Distribution of cross-section types in the y-direction (lengthwise)

Table 1 shows the first 6 natural frequencies of the train body for free-free BCs. Classical Timoshenko beam method (TBM) as well as up to the fifth-order TE refined train body models by the present DSM approach are given in Table 2.

Table 1 First to sixth natural frequencies (Hz) for the FF train body.

	Mode 1[a]	Mode 2[b]	Mode 3[c]	Mode 4[d]	Mode 5[e]	Mode 6[f]
N=5	50.3449	59.7189	82.2272	109.2521	127.8564	145.3032
N=4	50.9998	60.0692	82.4901	111.4041	129.5863	145.3141
N=3	51.2261	60.1823	86.9639	113.2928	130.6047	145.3141
N=2	53.2889	61.870	87.8691	126.1918	140.4748	145.3489
N=1	53.2628	61.8535	89.5141	104.9372	126.0787	140.4051
TBM	-	61.8535	-	-	125.0255	140.4051

[a] First flexural mode on plane *yz*. [b] First flexural mode on plane *xy*. [c] First torsional mode.
[d] Second flexural mode on plane *yz*. [e] Second flexural mode on plane *xy*. [f] Second torsional mode.

Fig.4 shows the first six modes of the train body for free-free BCs. The Mode 1 is first flexural mode on plane *yz*, the Mode 2 is first flexural mode on plane *xy*, the Mode 3 is first torsional mode, the Mode 4 is second flexural mode on plane *yz*, the Mode 5 is second flexural mode on plane *xy* and the Mode 6 is second torsional mode. Moreover, it has been demonstrated that CUF TE higher-order models can deal with non-classical phenomena such as torsion, shear effects and couplings. Train body elasticity solutions can, in fact, be reproduced with CUF models if a sufficient number of terms are considered in the kinematic field of the beam theory.

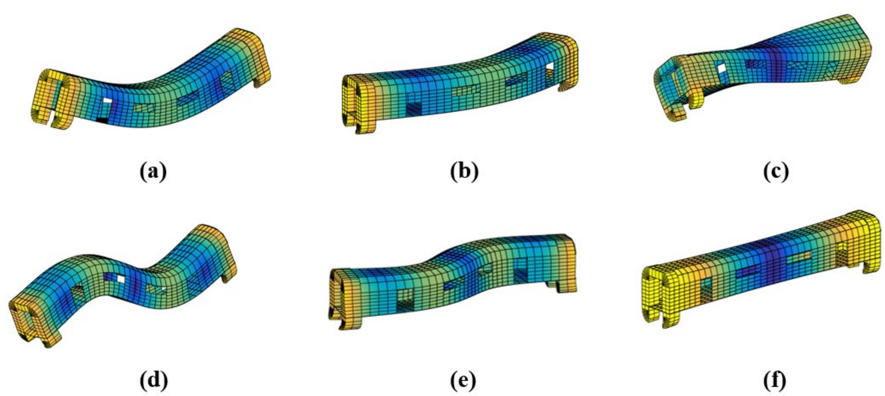


Fig.4 First flexural on plane yz (a), first flexural on plane xy (b), first torsional mode (c), second flexural on plane yz (d), second flexural mode on plane xy (e), second torsional mode (f) for the FF train body, N=4.

Conclusions

In the framework of CUF, the introduction of the parallel axis theorem in the Taylor expansion greatly improves its applicability: 1) geometrical boundary conditions can be applied in subdomains of the cross-section (and not only to the whole cross-section). 2) cross-sections can be divided into further beam sections and easily assembled. Combined with DSM, a high-order DS matrix is developed and the natural frequencies and mode shapes of the train body structure are calculated using the WW algorithm. Through further validation, the method can provide a powerful tool for the dynamic analysis and optimised design of laminated composite train body.

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