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Magnetoelastic deformation of conductive semilinear hyperelastic solids

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Abstract. This work investigates the radial deformation of conductive magneto-hyperelastic solid cylinder subject to azimuthal magnetic field. It shows effect of the current density on the radial deformation of the solid. A simple magnetoelastic energy function is proposed for the cylinder under consideration such that its purely elastic part corresponds to the strain energy of the well-known semilienar hyperelastic materials. The consequent magnetoelasticity field equations, in conjunction with the accompanying boundary conditions, are specialized for application to the problem of radial deformation of solid cylinder. The obtained magnetoelastic constitutive model shows that the stress distribution in the solids is sensitive to the magnetic induction while the associated magnetic field at point within the cylinder is deformation-dependent. Furthermore, it is revealed that the azimuthal magnetic induction produced by steady current within the solid cylinder increases along its radius. Finally, and among other things, the graphical illustration shows that the effect of steady axial current density on the magnitude of the displacement function at points within the cylinder is significantly pronounced.

Introduction

Conductive magnetoactive elastomers (CMEs) are magneto-sensitive materials that conduct electricity. They are manufactured by mixing micron/nano -size magnetic and conductive particles into nonmagnetic rubber-like matrices. CMEs exhibit change in mechanical response when subject to applied magnetic field and/ or current of electricity. The widespread applications of these materials have continued to instigate the needs for the development of new magnetoelasticity theories. In the fundamental formulation, magnetoelasticity field equations govern magneto-mechanical interaction of solids. These equations consist of magnetostatic and elasticity fields equations, and are used to construct solutions to problems involving magnetoelastic deformation.

Finite magnetoelastic interaction of solids has long been a subject of interest since the classic studies of Maugin [1], Eringen and Maugin [2] and Pao [3]. Recently, Pei et al. [4] investigated nonlinear magnetoelastic deformation of porous solids; Reddy and Saxena [5] studied instabilities in axisymmetric magnetoelastic deformation of a cylindrical membrane; Garcia-Gonzalez and Hossain [6] proposed a microstructural-based approach to model magneto-viscoelasticity materials at finite strain; Ren et al. [7] studied multi-functional soft-bodied jellyfish-like swimming; Bostola and Hossain [8] gave a review on magneto-mechanical charactirizations of magnetorheological elastomers; Dorfman and Ogden [9] studied nonlinear theory of electroelastic and magnetoelastic interactions; Nedjar [10] proposed a modelling framework for finite strain magnetoviscoelasticity; and Saxena et al. [11] developed a finite deformation theory for magneto viscoelasticity.

In view of Fadodun et al. [12], this work proposes a simple magnetoelastic energy function for conductive semilinear magnetohyperelastic solids. Using the laws of thermodynamic, Coleman-Noll procedure and tensor calculus, the study develops a magnetoelastic constitutive model for the solids under consideration. The consequent magnetoelasticity field equations together with the accompanying boundary conditions are specialized for applications to the problem of radial deformation of a conductive magnetoelastic cylinder subject to steady current of electricity. The rest of the paper is as follow: the first sections present magnetoelasticity field equations and constitutive relations while the remaining sections detail the solution to the radial deformation problem of magneto-sensitive solid cylinder subject to steady axial current density.

Kinematics

Consider a stress-free conductive magnetohyperelastic solid occupying the reference configuration $\Omega_0 \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega_0$ and surface outward unit normal vector $\vec{N}$. When subject to magnetic field and /or mechanical surface load the body deforms onto deformed configuration Ω with boundary $\partial\Omega$ and surface outward unit normal vector $\vec{n}$. The deformation of the body is defined by vector function $\vec{\varphi}$

$$\vec{\varphi}: \overline{\Omega}_0 \rightarrow \overline{\Omega}, \tag{1}$$

such that $\vec{x} = \vec{\varphi}(\vec{X})$ where $\vec{X}$ denotes position vector of material points in Ω_0 and $\vec{x}$ represents position vector of the corresponding material points in Ω. The closures Ω_0 and Ω in Eq. (1) are defined by

$$\overline{\Omega}_0 = \Omega_0 \cup \partial\Omega_0 \quad and \quad \overline{\Omega} = \Omega \cup \partial\Omega.$$

The deformation gradient $\boldsymbol{F}$ is defined by

$$\boldsymbol{F} = Grad\, \vec{x} = Grad\, \vec{\varphi}(\vec{X}), \tag{2}$$

where $Grad$ is the gradient operator with respect to Ω_0. At an arbitrary point $\vec{X}$, the determinant $\det(\boldsymbol{F}) > 0$ measures the local volume change.
Applying the polar decomposition theorem, the deformation gradient $\boldsymbol{F}$ is decomposed into product of second-rank tensors $\boldsymbol{O}^D$ and $\mathbf{U}$

$$\boldsymbol{F} = \boldsymbol{O}^D \boldsymbol{U}, \tag{3}$$

where $\boldsymbol{O}^D$ is the orthogonal rotation tensor and $\boldsymbol{U}$ is the right stretch symmetric tensor.
The tensors $\boldsymbol{U}$ and $\boldsymbol{O}^D$ are obtained by the relations

$$\boldsymbol{U} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}} = \sqrt{\boldsymbol{C}} \quad and \quad \boldsymbol{O}^D = \boldsymbol{F}\boldsymbol{U}^{-1}, \tag{4}$$

where $\boldsymbol{F}^T$ is the transpose of $\boldsymbol{F}$, $\boldsymbol{U}^{-1}$ is the inverse of $\boldsymbol{U}$ and $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy-Green deformation tensor [12].

Eulerian Form/Description: Magneto-Mechanical Field Equations

Let $\vec{H}, \vec{B}$ *and* $\vec{M}$ denote the Eulerian forms of the magnetic field, magnetic induction and effective magnetization vectors respectively. For a purely magnetostatic field produced by steady current, and in the absence of electric interaction and surface current the Maxwell's field equations read

$$curl\, \vec{H} = 4\pi\vec{J}, \quad div\, \vec{B} = 0, \tag{5}$$

where the operators div *and* $curl$ are defined in the deformed configuration Ω and $\vec{J}$ is the current density in Eulerian form. The current density $\vec{J}$ satisfies the equation

$$div\,\vec{J} = 0. \tag{6}$$

In magnetic materials, the vectors $\vec{H}, \vec{B}\ and\ \vec{M}$ are related by the constitutive law

$$\vec{B} = \mu\vec{H} = \mu_0(\overrightarrow{H} + \overrightarrow{M}), \tag{7}$$

where μ is the magnetic permeability of the material and μ_0 is the magnetic permeability of free space.
In free space exterior to the body, the corresponding magnetostatic fields are denoted by vectors $\vec{H}^*$ and $\vec{B}^*$, which are governed by the equations

$$curl\,\vec{H}^* = \vec{0},\ \ div\,\vec{B}^* = 0, \quad \vec{B}^* = \mu_0\vec{H}^*. \tag{8}$$

At the bounding surface of the considered material in the deformed configuration Ω, the standard boundary conditions associated with Eq. (5) are

$$\vec{n} \times \left(\vec{H}^* - \vec{H}\right) = \vec{0},\ \ \vec{n} \cdot \left(\vec{B}^* - \vec{B}\right) = 0 \qquad on \quad \partial\Omega, \tag{9}$$

where $\vec{n}$ is the unit outward normal vector on $\partial\Omega$.
Let $\boldsymbol{T}$ denote the total stress tensor which incorporates magnetostatic body forces. In the absence of mechanical body forces the mechanical equilibrium equation reads

$$div\,\boldsymbol{T} = \vec{0}. \tag{10}$$

The standard boundary condition accompanying the equilibrium equation is

$$\boldsymbol{T}\vec{n} = \vec{t}_a + \vec{t}_m \qquad on \quad \partial\Omega, \tag{11}$$

where $\vec{t}_a$ is the mechanical traction on $\partial\Omega$ per unit area, $\vec{t}_m = \boldsymbol{T}^*\vec{n}$ is the load due to the Maxwell stress

$$\boldsymbol{T}^* = \mu_0\left(\vec{H}^* \otimes \vec{H}^*\right) - \tfrac{1}{2}\mu_0(\vec{H}^* \cdot \vec{H}^*)\boldsymbol{I}, \tag{12}$$

and $\boldsymbol{I}$ is the second-order unit tensor in Ω [9, 12].

Lagrangian Form/Description: Magneto-Mechanical Field Equations
Let $\vec{H}_L,\ \vec{B}_L\ and\ \vec{M}_L$ denote the Lagrangian magnetic field, magnetic induction and effective magnetization vectors respectively. The Lagrangian variables $\vec{H}_L,\ \vec{B}_L\ and\ \vec{M}_L$ are related to the Eulerian quantities $\vec{H}, \vec{B}\ and\ \vec{M}$ by

$$\vec{H}_L = \boldsymbol{F}^T\vec{H},\ \vec{B}_L = \det(\boldsymbol{F})\,\boldsymbol{F}^{-1}\vec{B},\ \vec{M}_L = \boldsymbol{F}^T\vec{M}, \tag{13}$$

where the tensor $\boldsymbol{F}^{-1}$ is the inverse of $\boldsymbol{F}$ and $\det(\boldsymbol{F})$ is the determinant of $\boldsymbol{F}$.
The magnetostatic field equations in Lagrangian forms read

$$Curl\,\vec{H}_L = 4\pi\vec{J}_L, \qquad Div\,\vec{B}_L = 0, \tag{14}$$

where the operators $Div\ and\ Curl$ are defined in the reference configuration Ω_0, and $\vec{J}_L = \det(\boldsymbol{F})\,\boldsymbol{F}^{-1}\vec{J}$ is the Lagrangian current density satisfying the equation

$$Div\,\vec{J}_L = 0. \tag{15}$$

Similarly, the vectors Let $\vec{H}_L,\ \vec{B}_L\ and\ \vec{M}_L$ are related by

$$\vec{B}_L = \mu_0 \det(\boldsymbol{F})\boldsymbol{C}^{-1}(\vec{H}_L + \vec{M}_L). \tag{16}$$

In addition, the vectors $\vec{H}_L\ and\ \vec{B}_L$ satisfy the standard boundary conditions

$$\vec{N} \times \left(\overrightarrow{\boldsymbol{F}^T H^*} - \vec{H}_L\right) = \vec{0}, \quad \vec{N} \cdot \left(\det(\boldsymbol{F})\,\boldsymbol{F}^{-1}\vec{B}^* - \vec{B}_L\right) = 0 \quad on \quad \partial\Omega_0. \tag{17}$$

Let $\boldsymbol{P}$ denote the total first Piola-Kirchhoff's stress tensor. The total stress tensor $\boldsymbol{T}$ and first Piola-Kirchhoff's stress tensor $\boldsymbol{P}$ are related by

$$\boldsymbol{P} = \det(\boldsymbol{F})\,\boldsymbol{T}\boldsymbol{F}^{-T}, \tag{18}$$

where $\boldsymbol{F}^{-T}$ is the inverse of $\boldsymbol{F}^T$.
In term of $\boldsymbol{P}$ the equilibrium equation assumes the equivalent form

$$Div\,\boldsymbol{P} = \vec{0}. \tag{19}$$

The corresponding boundary condition reads

$$\boldsymbol{P}\vec{N} = \vec{t}_F + \vec{t}_{mF} \quad on \quad \partial\Omega_0, \tag{20}$$

where $\vec{t}_F$ is the mechanical traction on $\partial\Omega_0$ per unit area, $\vec{t}_{mF} = \boldsymbol{P}^*\vec{N}$ and $\boldsymbol{P}^* = \boldsymbol{P} = \det(\boldsymbol{F})\,\boldsymbol{T}^*\boldsymbol{F}^{-T}$ is the pull back version of the Maxwell stress $\boldsymbol{T}^*$ [12].

Magnetoelastic Energy Function and Constitutive Model

In order to complete the mathematical equations formulation for the study, we choose the deformation gradient $\boldsymbol{F}$ and magnetic induction vector $\vec{B}_L$ as independent variables; and model magnetoelastic constitutive laws that give first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and magnetic field vector $\vec{H}_L$ in terms of $\boldsymbol{F}$ and $\vec{B}_L$. Consequently, we take the magnetoelastic Helmholtz free energy function $\Phi = \Phi(\boldsymbol{F}, \vec{B}_L)$ to depend on $\boldsymbol{F}$ and $\vec{B}_L$, and ensure the objectivity condition

$$\Phi(\mathbf{Q}\boldsymbol{F}, \vec{B}_L) = \Phi(\boldsymbol{F}, \vec{B}_L),$$

is satisfied for all proper orthogonal second-rank tensor $\mathbf{Q}$.
Using the laws of thermodynamics and Coleman-Noll procedure, the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the Lagrangian magnetic field vector $\vec{H}_L$ are obtained through the relations

$$\boldsymbol{P} = \frac{\partial\Phi(\boldsymbol{F}, \vec{B}_L)}{\partial \boldsymbol{F}}, \qquad \vec{H}_L = \frac{\partial\Phi(\boldsymbol{F}, \vec{B}_L)}{\partial \vec{B}_L}. \tag{21}$$

Now, we recall and state the elastic strain energy function Φ^* per unit volume

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$$\Phi^*(\boldsymbol{F}) = \frac{1}{2}\lambda_e \mathbb{I}_1^2(\boldsymbol{U}-\boldsymbol{I}_0) + \mu_e \mathbb{I}_2(\boldsymbol{U}-\boldsymbol{I}_0), \quad (22)$$

for an isotropic semilinear hyperelastic solid, where $\mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0)$ is the first invariant of the second-rank tensor $(\boldsymbol{U}-\boldsymbol{I}_0)$, $\mathbb{I}_2(\boldsymbol{U}-\boldsymbol{I}_0) = \mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0)^2$, λ_e, μ_e are the Lame's constants and $\boldsymbol{I}_0$ is the second-rank unit tensor in the reference configuration [12]

Following Fadodun el al. [12], Melnikov and Ogden [13] and Dorfmann and Ogden [14], we generalize and consider a simple energy function of the form

$$\Phi(\boldsymbol{F}, \vec{B}_L) = \frac{1}{2}\lambda_e \mathbb{I}_1^2(\boldsymbol{U}-\boldsymbol{I}_0) + \mu_e \mathbb{I}_2(\boldsymbol{U}-\boldsymbol{I}_0) + \frac{1}{2\mu}\vec{B}_L \cdot \boldsymbol{U} \cdot \vec{B}_L, \quad (23)$$

for the magnetoelastomeric solid under consideration such that its purely elastic part corresponds to the semilinear hyperelastic energy function in Eq. (22), where the scalar μ is the permeability of the solid.

The Frechet derivatives of invariants $\mathbb{I}_1(\boldsymbol{U}-)^2$ and $\mathbb{I}_1^2(\boldsymbol{U}-\boldsymbol{I}_0)$ with respect to $\boldsymbol{F}$ are [12]

$$\frac{\partial \mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0)^2}{\partial \boldsymbol{F}} = 2(\boldsymbol{U}-\boldsymbol{I}_0)\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{F}} = 2(\boldsymbol{U}-\boldsymbol{I}_0)\boldsymbol{O}^{DT} = 2(\boldsymbol{F}^T - \boldsymbol{O}^{DT}), \quad (24)$$

and

$$\frac{\partial \mathbb{I}_1^2(\boldsymbol{U}-\boldsymbol{I}_0)}{\partial \boldsymbol{F}} = 2\mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0)\boldsymbol{I}_0\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{F}} = 2\mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0)\boldsymbol{I}_0\boldsymbol{O}^{DT} = 2\mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0)\boldsymbol{O}^{DT}, \quad (25)$$

respectively. The tensors $\boldsymbol{O}^{DT}$ and $\boldsymbol{F}^T$ are transposes of $\boldsymbol{O}^D$ and $\boldsymbol{F}$.
Next, the Fréchet derivative of invariant $\vec{B}_L \cdot \boldsymbol{U} \cdot \vec{B}_L$ with respect to $\boldsymbol{F}$ is

$$\frac{\partial \vec{B}_L \cdot \boldsymbol{U} \cdot \vec{B}_L}{\partial \boldsymbol{F}} = (\vec{B}_L \otimes \vec{B}_L)\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{F}} = (\vec{B}_L \otimes \vec{B}_L)\boldsymbol{O}^{DT}. \quad (26)$$

In view of Eqs. (24)-(26), and substituting Eq. (23) into Eq. (21) gives the total first Piola-Kirchhoff's stress tensor $\boldsymbol{P}$

$$\boldsymbol{P} = \frac{\partial \Phi(\boldsymbol{F}, \vec{B}_L)}{\partial \boldsymbol{F}} = 2\mu_e \boldsymbol{F}^T + \left((\lambda_e \mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0) - 2\mu_e)\boldsymbol{I}_0 + \frac{1}{2\mu}(\vec{B}_L \otimes \vec{B}_L)\right)\boldsymbol{O}^{DT}, \quad (27)$$

and deformation-dependent magnetic field vector $\vec{H}_L$

$$\vec{H}_L = \frac{\partial \Phi(\boldsymbol{F}, \vec{B}_L)}{\partial \vec{B}_L} = \frac{1}{\mu}\boldsymbol{U}\vec{B}_L, \quad (28)$$

as the magnetoelastic constitutive model for the magnetoelastic solids under consideration, where ⊗ denotes the tensor product.
In view of Eq. (18), the corresponding Eulerian total stress tensor $\boldsymbol{T}$ is

$$\boldsymbol{T} = (\det(\boldsymbol{F}))^{-1}\left(2\mu_e \boldsymbol{F}^{2T} + \left((\lambda_e \mathbb{I}_1(\boldsymbol{U}-\boldsymbol{I}_0) - 2\mu_e)\boldsymbol{I}_0 + \frac{1}{2\mu}(\vec{B}_L \otimes \vec{B}_L)\right)(\boldsymbol{F}\boldsymbol{O}^D)^T\right).$$

Remark 1: The obtained magnetoelastic constitutive model in Eqs. (27) and (28) shows that the stress distribution is sensitive to the magnetic induction generated while the magnetic field at point within the body is deformation-dependent.

Remark 2: In the absence of externally applied magnetic field, the magnetic induction vector varnishes at points within the solids, and the derived constitutive equations degenerate to $\boldsymbol{P} = 2\mu_e \boldsymbol{F}^T + (\lambda_e \mathbb{I}_1(\boldsymbol{U} - \boldsymbol{I}_0) - 2\mu_e)\boldsymbol{O}^{DT}$ and $\vec{H}_L = \vec{0}$ which yield purely mechanical stress in the solids [12]. In addition, if the stretch symmetric tensor $\boldsymbol{U} = \boldsymbol{I}_0$ the derived model degenerates to $\boldsymbol{P} = \frac{1}{2\mu_L}\left(\vec{B}_L \otimes \vec{B}_L\right)\boldsymbol{O}^{DT}$ and $\vec{H}_L = \frac{1}{\mu}\vec{B}_L$ which implies that the body exhibits purely magnetic behaviour.

Application: Magnetoelastic Deformation of Conductive Hyperelastic Cylinder

In view of the constitutive Eqs. (27) and (28), it is convenient to solve the problem of the radial deformation of conductive semilinear magneto-hyperelastic solid cylinder in the Lagrangian frame of reference. The theory of magnetoelastcity presented in the previous sections is now specialized for application to the problem of magnetoelastic deformation of a solid cylinder. The cylinder under consideration has radius A and is subject to uniform axial current density. The geometry of the cylinder is assumed to be sufficiently long/thin such that the edge effect is neglected.

Let the cylindrical coordinates (R, Θ, Z) with associated unit basis vectors $\vec{E}_R, \vec{E}_\Theta, \vec{E}_Z$ describe the position vector $\vec{R} = R\vec{E}_R + Z\vec{E}_Z$ of material point of the cylinder in the reference configuration defined by

$$0 \leq R \leq A, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \tag{29}$$

where A and L are the radius and length of the cylinder respectively.

Invoking the constraint of circular symmetry, and let the cylindrical coordinates (r, θ, z) with unit basis vectors $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$ give the position vector $\vec{r} = r\vec{e}_r + z\vec{e}_z$ of the corresponding material point in the deformed configuration, the deformation of the cylinder is defined by

$$r = r(R), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{30}$$

where $r(R)$ is a function of R only and λ_z is the uniform axial stretch.

Using Eqs. (2) and (30), the deformation gradient $\boldsymbol{F}$ is

$$\boldsymbol{F} = Grad\,\vec{r} = \frac{\partial r}{\partial R}\vec{e}_r \otimes \vec{E}_R + \frac{r}{R}\vec{e}_\theta \otimes \vec{E}_\Theta + \frac{\partial z}{\partial Z}\vec{e}_z \otimes \vec{E}_Z,$$

$$\boldsymbol{F} = Grad\,\vec{r} = \frac{\partial r}{\partial R}\vec{e}_r \otimes \vec{E}_R + \frac{r}{R}\vec{e}_\theta \otimes \vec{E}_\Theta + \lambda_z \vec{e}_z \otimes \vec{E}_Z, \tag{31}$$

where $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$ and $\vec{E}_R, \vec{E}_\Theta, \vec{E}_Z$ are the orthonormal basis vectors in Ω and Ω_0 respectively.

Using Eqs. (3), (4) and (31) gives

$$(\boldsymbol{U})_{ij} = (\boldsymbol{F})_{ij} \ and \ (\boldsymbol{O}^D)_{ij} = (\boldsymbol{I}_0)_{ij}, \tag{32}$$

where for any second-rank tensor $\boldsymbol{A}$, $(\boldsymbol{A})_{ij}$ denotes the components of $\boldsymbol{A}$.

Solution of magnetostatic field equations

Let the axis of the solid cylinder of radius A and constant conductivity σ be taken along the Z axis and let $\vec{J}_L = J_L\vec{E}_Z$be the uniform axial current density along the axis of the cylinder, where J_L is the magnitude of $\vec{J}_L$ and $\vec{E}_Z$ is the unit vector along the axis of the cylinder.

For this problem, Eq. (15) is satisfied for uniform $\vec{J}_L$. The solution of Eq. 14(b) is obtained by introducing a uniquely defined vector (magnetic vector potential) $\vec{G}_L$ such that

$$\vec{B}_L = Curl\ \vec{G}_L,\ Div\ \vec{G}_L = 0. \tag{33}$$

Using Eq. 33(b) and $\vec{B}_L = \mu\vec{H}_L$ in Eq. 14(a) gives

$$Curl\left(Curl\ \vec{G}_L\right) = \begin{cases} 4\pi\mu J_L\vec{E}_Z, & R \le A \\ \vec{0}, & R > A \end{cases}. \tag{34}$$

The form of Eq. (34) suggests that $\vec{G}_L = G(R,\Theta,Z)\vec{E}_Z$ where $G(R,\Theta,Z)$ is a function of cylindrical coordinates $R,\ \Theta\ and\ Z$. Meanwhile Eq. 33(b) shows that G is independent of Z and by symmetry, G is independent of Θ, thus, $\vec{G}_L = G(R)\vec{E}_Z$ is a function of R only [15].

The resolute of $\left(Curl\ \vec{G}_L\right)$ and $Curl\left(Curl\ \vec{G}_L\right)$ are

$$\begin{cases} Curl\ \vec{G}_L = \left(0, -\frac{dG(R)}{dR}, 0\right) \\ Curl\left(Curl\ \vec{G}_L\right) = \left(0, 0, -\frac{1}{R}\frac{d}{dR}\left(R\frac{dG(R)}{dR}\right)\right) \end{cases}. \tag{35}$$

Substituting Eq. 35(b) into Eq. (34) gives

$$\begin{cases} \frac{1}{R}\frac{d}{dR}\left(R\frac{dG(R)}{dR}\right) + 4\pi\mu J_L, & R \le A \\ \frac{1}{R}\frac{d}{dR}\left(R\frac{dG(R)}{dR}\right) = 0, & R > A \end{cases}. \tag{36}$$

The solution of Eq. (36) yields

$$G(R) = \begin{cases} C_1 \ln R + C_2 - \pi\mu J_L R^2, & R \le A \\ C_3 \ln R + C_4, & R > A \end{cases}, \tag{37}$$

where $C_i, i = 1,2,3,4$ are constants to be determined.

Since $G(R)$ must be finite along the axis of the tube ($R = 0$), $C_1 = 0$. Thus,

$$G(R) = \begin{cases} C_2 - \pi\mu J_L R^2, & R \le A \\ C_3 \ln R + C_4, & R > A \end{cases}. \tag{38}$$

The constant C_3 is obtained by using the Maxwell's first circuital relation

$$\int \vec{H}\cdot d\vec{s} = 4\pi I \Longrightarrow \frac{1}{\mu_L}\int_0^{2\pi} -\frac{dG(R)}{dR} R d\Theta = 4\pi I, \tag{39}$$

where I is the current flowing in the tube. Hence,

$$C_3 = -2\pi \mu J_L A^2. \tag{40}$$

Substituting Eq. (40) into Eq. (38) gives

$$G(R) = \begin{cases} C_2 - \pi \mu J_L R^2, & R \leq A \\ -2\pi \mu J_L A^2 \ln R + C_4, & R > A \end{cases}. \tag{41}$$

Recall that the magnetic vector potential is continuous at the surface of seperation, thus,

$$C_2 - \pi \mu J_L A^2 = -2\pi \mu J_L A^2 \ln A + C_4. \tag{42}$$

and setting $C_4 = 0$ (without loss of generality) gives

$$C_2 = \pi \mu J_L A^2 - 2\pi \mu J_L A^2 \ln A. \tag{43}$$

Substituting Eq. (43) into Eq. (41) gives the solution

$$G(R) = \begin{cases} \pi \mu J_L (A^2 - R^2) - 2\pi \mu J_L A^2 \ln A, & R \leq A \\ -2\pi \mu J_L A^2 \ln R, & R > A \end{cases}. \tag{44}$$

In view of Eqs. 33(a) and 35(a), the magnetic induction $\vec{B}_L = \left(0, -\frac{dG(R)}{dR}, 0\right) = (0, B_\Theta, 0)$ has non-varnishing azimuthal resolute [15]

$$B_\Theta = -\frac{dG(R)}{dR} = \begin{cases} 2\pi \mu J_L R, & R \leq A \\ \frac{2\pi \mu J_L A^2}{R}, & R > A \end{cases}. \tag{45}$$

Using Eqs. (28), (31), (32) and (45), the corresponding magnetic field strength $\vec{H}_L = (0, H_\Theta, 0)$ within the tube has non-varnishing azimuthal component H_Θ

$$H_\Theta = \frac{1}{\mu} \frac{r(R)}{R} B_\Theta = 2\pi J_L r(R), \tag{46}$$

where $r(R)$ is a function of R only.

Solution of equilibrium equation

Using Eqs. (31), (32) and 45(a) in Eq. (27) gives the non-zero components $P_{RR}, P_{\Theta\Theta}, P_{ZZ}$ of the first Piola-Kirchhoff's stress tensor $\boldsymbol{P}$:

$$P_{RR} = 2\mu_e \left(\frac{dr}{dR} - 1\right) + \lambda_e \left(\frac{dr}{dR} + \frac{r}{R} + \lambda_z - 3\right), \tag{47}$$

$$P_{\Theta\Theta} = 2\mu_e \left(\frac{r}{R} - 1\right) + \lambda_e \left(\frac{dr}{dR} + \frac{r}{R} + \lambda_z - 3\right) + 2\mu (\pi J_L R)^2, \tag{48}$$

$$P_{ZZ} = 2\mu_e (\lambda_z - 1) + \lambda_e \left(\frac{dr}{dR} + \frac{r}{R} + \lambda_z - 3\right). \tag{49}$$

Using Eqs.(47)-(49), the equilibrium equation in Eq. (19) reduces to

$$\frac{dP_{RR}}{dR} + \frac{1}{R}(P_{RR} - P_{\Theta\Theta}) = 0. \quad (50)$$

Substituting Eqs. (47)-(48) into Eq. (50) gives

$$\left(\frac{d^2r}{dR^2} + \frac{1}{R}\frac{dr}{dR} - \frac{r}{R^2}\right) + \left(\frac{2\mu\pi^2 J_L^2}{2\mu_e + \lambda_e}\right) R = 0. \quad (51)$$

The solution of Eq. (51) gives the deformation function

$$r(R) = C_5 R + \frac{C_6}{R} + \varpi_J R^3, \quad (52)$$

where $C_5,\ C_6$ are constants and the scalar $\varpi_J = \frac{1}{4}\left(\frac{\mu\pi^2 J_L^2}{2\mu_e + \lambda_e}\right)$ depends on the axial current density.
The displacement field u(R) at points within the tube is defined by

$$u(R) = r(R) - R = C_5^* R + \frac{C_6}{R} + \varpi_J R^3, \quad (53)$$

where the constant $C_5^* = C_5 - 1$.
Since the displacement u(R) must be finite at $R = 0$, then $C_6 = 0$. Furthermore, using the condition $P_{RR} = 0$ at the tube surface $R = A$yields
$C_5^* = \left(\frac{3\mu_e + 2\lambda_e}{\mu_e + \lambda_e}\right)\varpi_J A^2 - \left(\frac{\lambda_e}{2(\mu_e + \lambda_e)}\right)(\lambda_z - 1),$
where λ_z is the uniform axial stretch.
Thus, the displacement $u(R)$ at point within the tube

$$u(R) = \varpi_J R^3 + \left(\frac{3\mu_e + 2\lambda_e}{\mu_e + \lambda_e}\right)\varpi_J A^2 R - \left(\frac{\lambda_e}{2(\mu_e + \lambda_e)}\right)(\lambda_z - 1)R. \quad (54)$$

Using Eqs. (46), (54) and knowing that $r(R) = u(R) + R$, the magnetic field strength H_Θ at points within the tube is

$$H_\Theta = 2\pi J_L \left(\varpi_J R^3 + \left(\frac{3\mu_e + 2\lambda_e}{\mu_e + \lambda_e}\right)\varpi_J A^2 R - \left(\frac{\lambda_e}{2(\mu_e + \lambda_e)}\right)(\lambda_z - 1)R + R\right). \quad (55)$$

In the absence of current density J_L (i.e. $J_L = 0$), the displacement function u(R) and magnetic field function H_Θ in Eqs. (54) and (55) reduce/degenerate to

$$u(R) = \left(\frac{\lambda_e}{2(\mu_e + \lambda_e)}\right)(1 - \lambda_z)R,\ H_\Theta = 0, \quad (56)$$

respectively.

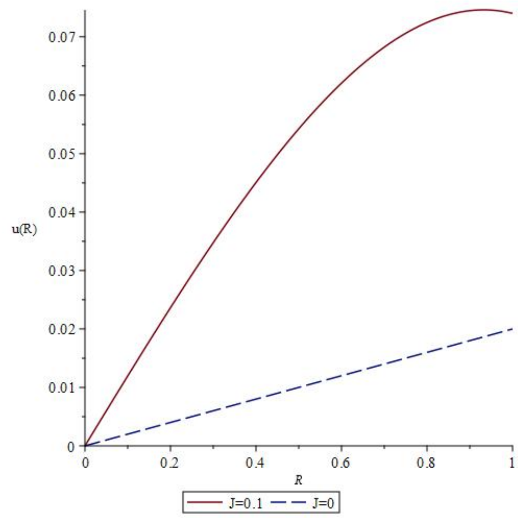

Fig. 1: This plot shows that the effect of steady current density on the magnitude of displacement function at point within solid cylinder is significantly pronounced.

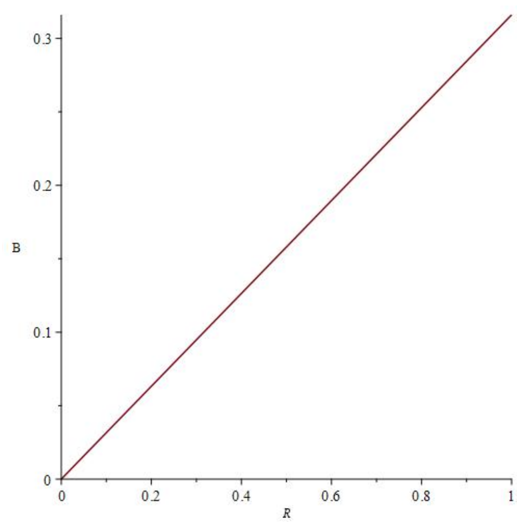

Fig. 2: This plot shows that the azimuthal magnetic induction produced by the steady current within the solid cylinder increases linearly along the radius of the cylinder.

Conclusion

The study develops a new magnetoelastic constitutive theory for modelling magneto-mechanical interaction of solids. The theory is specialized for application to the problem of radial deformation of a solid circular cylindrical made of conductive semilinear magnetohyperelastic materials. It is obtained that the stress propagation in the solid cylinder is sensitive to the magnetic induction produced by uniform axial current density while the associated magnetic field is deformation-dependent. Furthermore, it is shown that the effect of uniform axial current density on the deformation of the tube is significantly pronounced. Finally, the results in this study find applications in design of soft actuators, sensors and energy harvesters to mention a few.

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