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Dynamic deformation analysis of a spherical cavity explosion

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Abstract. The problem of rapid explosion of a spherical cavity in an infinite elastic media of Achenbach and Sun, Israilov and Hamidou is revisited in this study under central symmetric considerations. The governing partial differential equation is reformulated as a differential-difference equation, with compatibility conditions for the unknown cavity's radial evolution, with the unknown displacement being the exponential generating function of a formal radial Laurent series. It is shown that for cavity explosion at constant speed, the modified problem admits self-similar displacement profiles of the inverse hyperbolic type for conjugate decreasing cavity pressures inversely proportional to the time.

Introduction

This short paper provides an effective exact method of lines for the dynamic response of a rapidly expanding surface in an infinite elastic space consisting of a linear homogeneous and isotropic Hookean material. The mechanical problem is that of the determination of the deformation ensuing exterior to a rapidly expanding spherical cavity in an infinite elastic media. The transition into anelastic constituent properties is excluded from this consideration, and only infinitesimal strain is assumed in the media which is also assumed homogeneous.

Moreover, the mathematical problem is that of the simultaneous determination of the displacement field in the region exterior to that cavity and the dynamics of the exploding (or rapidly evolving) surface in the time. Further simplifying assumptions are that the process is isothermal and the central symmetry is invoked within a spherical polar system of coordinates. More precisely, the constitutive relation of the material is given by

$$\tilde{\sigma} = \lambda \operatorname{tr}(\tilde{\varepsilon})\tilde{\mathrm{I}} + 2\mu\tilde{\varepsilon},$$

where $\tilde{\sigma}$ is the Cauchy stress tensor; $\tilde{\varepsilon}$ is the linearized Green-St.Venant strain tensor; $\tilde{\mathrm{I}}$ is the unit second rank tensor; and λ, μ are the Lame viscosity constants. It should be emphasized also that the strain considered here is infinitesimal.

Given the more recent interesting consideration by Israilov and Hamidou [1] in which a thorough review of the earlier work by Achenbach and Sun [2] is presented, we have decided to give yet a different view of the same problem by using a method of lines and thereby obtaining exact explicit solutions for a rapidly exploding surface moving at constant speed. It should be noted that in the exposition herein, the initial displacement is taken as non-zero and radially dependent, no *a priori* assumption of the nature of the evolution of the surface, and neither is the shape of the pressure in the cavity aforehand specified, the intent being for the compatible trio of displacement, surface evolution, and cavity pressure to naturally present itself through the analytical manipulations.

This study is by no means exhaustive, and has the limitations of being able to address only a class of aforestated compatible *trio*, but it has the strength of being able to give explicit solutions which offer the utility of being a benchmark for numerical and analytical solutions for problems of this class. Specifically, the problem in context is governed by the following partial differential equation, and attendant initial and boundary conditions:

$$\frac{\partial^2 u(r,\bar{t})}{\partial r^2}+\frac{2}{r}\frac{\partial u(r,\bar{t})}{\partial r}-\frac{2}{r^2}u(r,\bar{t})=\frac{1}{c^2}\frac{\partial^2 u(r,\bar{t})}{\partial \bar{t}^2} \qquad r>s(\bar{t})>0, \bar{t}>0; \qquad (2)_1$$

$$u(r,0)=\omega r, \ \frac{\partial u(r,\bar{t})}{\partial \bar{t}}\Big|_{\bar{t}=0}=0 \quad r>s(0)>0; \qquad (2)_2$$

$$\left(\frac{\partial u(r,\bar{t})}{\partial r}+\frac{2\nu}{1-\nu}\frac{u(r,\bar{t})}{r}\right)\Big|_{r=s(\bar{t})}=-\frac{1}{\rho c^2}\mathsf{P}(\bar{t}), \qquad (2)_3$$

where $u(r,\bar{t})$ is the radial displacement field in time $\bar{t}$; $c=\sqrt{\frac{\lambda+2\mu}{\rho}}$; ν the Poisson ratio of the body; and $\mathsf{P}(\bar{t})$ is time-dependent pressure in the cavity $r<s(\bar{t})$.

Despite the innocuous outlook of the problem, it is observed that there are inherent nonlinearities in the formulation of the problem, essentially being the moving surface $r=s(\bar{t})$. The current literature is replete with techniques for handling formulations of this kind, and it should be noted that in spite, problems of this class do not readily yield themselves to exact solutions.

Differential-Difference Reformulation

Motivated by the solution of the quasistatic state equation $(2)_1$

$$u(r,\bar{t})=\frac{c_1(\bar{t})}{r^2}+c_2(\bar{t})r \qquad (3)$$

we assume the displacement evolution of the elastic media in the region $r>s(\bar{t})$ which is ahead of the rapidly expanding surface in the form of a spatial Laurent series

$$u(r,t)=\sum_{j=0}^{\infty}\mathsf{M}_j(t)\frac{r^{-j}}{\Gamma(j+1)}, \qquad (4)$$

where $t:=c\bar{t}$. By elementary calculations, we have the representations

$$\frac{\partial^2 u(r,\bar{t})}{\partial r^2}+\frac{2}{r}\frac{\partial u(r,\bar{t})}{\partial r}-\frac{2}{r^2}u(r,\bar{t})\equiv\sum_{j=0}^{\infty}(j-2)(j+1)\mathsf{M}_j(t)\frac{r^{-(j+2)}}{\Gamma(j+1)};$$

$$\frac{1}{c^2}\frac{\partial^2 u(r,\bar{t})}{\partial \bar{t}^2}\equiv\ddot{\mathsf{M}}_0(t)+\ddot{\mathsf{M}}_1(t)r^{-1}+\sum_{j=0}^{\infty}\ddot{\mathsf{M}}_j(t)\frac{r^{-(j+2)}}{\Gamma(j+3)}. \qquad (5)$$

From Eq. (5) it is further observed that $\mathsf{M}_j(t)$ verifies the differential-difference equation

$$\begin{cases}\ddot{\mathsf{M}}_{j+2}(t)=(j^2-4)(j+1)^2\mathsf{M}_j(t);\\ \ddot{\mathsf{M}}_0(t)=0=\ddot{\mathsf{M}}_1(t),\end{cases} \qquad (6)$$

with the boundary condition of the problem suggesting the shape of $M_j(t)$ in the form

$$\mathsf{M}_j(t) \propto \mathsf{P}(t) s^{j+1}(t), \tag{7}$$

while, defining $s_0 := s(0)$, the zero initial conditions enforce the following constraints

$$\begin{cases} 0 < \sum_{j=0}^{\infty} \frac{\mathsf{M}_j(0)}{\Gamma(j+1)} s_0^{-j}; and \\ 0 < \sum_{j=0}^{\infty} \frac{\dot{\mathsf{M}}_j(0)}{\Gamma(j+1)} s_0^{-j}. \end{cases} \tag{8}$$

Solution to differential-difference equation

Assuming that $\mathsf{M}_j(t) = \mathsf{P}(t) s^{j+1}(t) m_j$, m_j a yet-to-be-determined sequence, the constraints in Eq. $(6)_2$ imply that

$$\begin{cases} \left(\ddot{\mathsf{P}}(t) s(t) + 2\dot{\mathsf{P}}(t)\dot{s}(t) \right) m_0 = 0; \\ \left(\ddot{\mathsf{P}}(t) s^2(t) + 4\dot{\mathsf{P}}(t)\dot{s}(t)s(t) + 2\dot{s}^2(t)\mathsf{P}(t) \right) m_1 = 0, \end{cases} \tag{9}$$

from which it is observed that for non-vanishing m_0 and m_1, the admissible surface evolution pattern and the relation

$$s(t) = \tau t + s_0; \mathsf{P}(t) = \beta s^{-1}(t), \tag{10}$$

$\beta \in \mathbb{R}$ obtains.

Correspondingly, Eq. $(6)_1$ yields the following equations

$$\begin{cases} \left(\ddot{\mathsf{P}}(t) s^{j+3}(t) + 2\dot{\mathsf{P}}\,\dot{s}(t) s^{j+2}(t)(j+3) + (j+2)(j+3)\mathsf{P}(t)\dot{s}^2(t)s^{j+1}(t) \right) m_{j+2} \\ = (j^2-4)(j+1)^2 \mathsf{P}(t) s^{j+1}(t) m_j; \end{cases} \tag{11$_1$}$$

$$\begin{cases} \left(\ddot{\mathsf{P}}(t) s^2(t) + 2\dot{\mathsf{P}}(t)\dot{s}(t)s(t)(j+3) + (j+2)(j+3)\mathsf{P}(t)\dot{s}^2(t) \right) m_{j+2} \\ = (j^2-4)(j+1)^2 \mathsf{P}(t) m_j. \end{cases} \tag{11$_2$}$$

Resulting from the relation $\ddot{\mathsf{P}}(t)s(t) + 2\dot{\mathsf{P}}(t)\dot{s}(t) = 0$, Eq. $(11)_2$ in turn yields

$$\left(2\dot{\mathsf{P}}(t)\dot{s}(t)s(t)(j+2) + (j+2)(j+3)\mathsf{P}(t)\dot{s}^2(t) \right) = (j^2-4)(j+1)^2 \mathsf{P}(t) s^{j+1}(t) m_j. \tag{12}$$

Finally, upon a reflection of Eq. (10) on Eq. (12), the governing equation for the sequence $\{m_j\}_0^\infty$ is obtained as

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$$\tau^2 m_{j+2} = (j-2)(j+1)m_j,\ j \ge 0, \tag{13}$$

which has the solution

$$m_j = \begin{cases} m_0 & j=0 \\ m_1 & j=1 \\ -\dfrac{2m_0}{\tau^2} & j=2, \\ -\dfrac{(j-1)}{2}\Gamma(j-2)\tau\left(\left(\dfrac{1}{\tau}\right)^j - \left(-\dfrac{1}{\tau}\right)^j\right)m_1 & j \ge 3 \end{cases} \tag{14}$$

with m_0, m_1 and τ being undetermined constants.

Convergence analysis

From difference equation (13), one can make some *a priori* convergence analysis of series (4). Firstly, it is observed that for $j \ge 3$,

$$\tau^2 \left|\frac{m_{j+2}}{m_{j+1}}\right|\left|\frac{m_{j+1}}{m_j}\right| = |(j-2)(j+1)| \tag{15}$$

from which it is further inferred that

$$\lim_{j\to\infty}\left|\frac{m_{j+1}}{m_j}\right| = \infty. \tag{16}$$

Moreover,

$$\lim_{j\to\infty}\frac{1}{|(j+1)|}\left|\frac{m_{j+1}}{m_j}\right| = |\tau|^{-1}. \tag{17}$$

By the ratio test, the region of convergence of series (4) is that which is determined, per the time t, by the limiting inequality

$$\left|\frac{s(t)}{r}\right| \times \lim_{j\to\infty}\left|\frac{m_{j+1}}{(j+1)m_j}\right| < 1, \tag{18}$$

which, by Eq.(17), is $r > |\tau|^{-1}\, s(t)$.

Explicit Solutions

In this section, we shall exploit the derived solution of the differential-difference equation, up to arbitrary constants, to explicitly determine the displacement evolution in the region exterior to the rapidly expanding surface.

The displacement of the spherical body ahead of the rapidly expanding surface, therefore, has an evolution of the form

$$u(r,t)=\beta m_0-\beta\frac{(s_0+\tau t)}{r}m_1-\beta\frac{(s_0+\tau t)^2}{\tau^2 r^2}m_0+ \\ +\beta\sum_{j=3}^{\infty}\frac{\tau m_1}{2j(j-2)}\left(\left(-\frac{1}{\tau}\right)^j-\left(\frac{1}{\tau}\right)^j\right)\left(\frac{s_0+\tau t}{r}\right)^j; \tag{19$_1$}$$

$$u(r,t)=\left(1-\left(\frac{s_0+\tau t}{\tau r}\right)^2\right)\beta m_0+ \\ +\frac{\beta}{2\tau r^2}\left(\tau r(s_0+\tau t)-(s_0+\tau(t-r))(s_0+\tau(t+r))\mathrm{ArcCoth}\left(\frac{\tau r}{s_0+\tau t}\right)\right)m_1. \tag{19$_2$}$$

Converting the temporal variable back to $\bar{t}$ gives the solution as

$$u(r,\bar{t})=\left(1-\left(\frac{s_0+\tau c\bar{t}}{\tau r}\right)^2\right)\beta m_0+ \\ +\frac{\beta}{2\tau r^2}\left(\tau r(s_0+\tau c\bar{t})-(s_0+\tau(c\bar{t}-r))(s_0+\tau(c\bar{t}+r))\mathrm{ArcCoth}\left(\frac{\tau r}{s_0+\tau c\bar{t}}\right)\right)m_1. \tag{20}$$

Initial conditions

Firstly, we take into account the zero initial speed condition $\frac{\partial u(r,0)}{\partial \bar{t}}=0$ for Eq. (20) to realize that $m_0=0$ simultaneously with the compatibility condition

$$r\tau-s_0\mathrm{ArcCoth}\left(\frac{\tau r}{s_0}\right)=0. \tag{21}$$

Set $\alpha:=\frac{r}{s_0}>1$. One observes that the relationship between the dimensionless parameters α and τ through the transcendental equation (21) is approximately

$$\alpha\tau\left(:=\frac{r}{s_0}\tau\right)\approx 1.199678. \tag{22}$$

This further constrains τ in the interval $(0,\infty)$ given the positivity of α. Noting the range of admissibility of τ, Fig.1 below gives a graphical relationship between α and τ. Secondly, an application of the initial displacement condition $u(r,0)=\omega r$ reveals that $m_1=\frac{2s\omega}{\beta\tau^2}$.

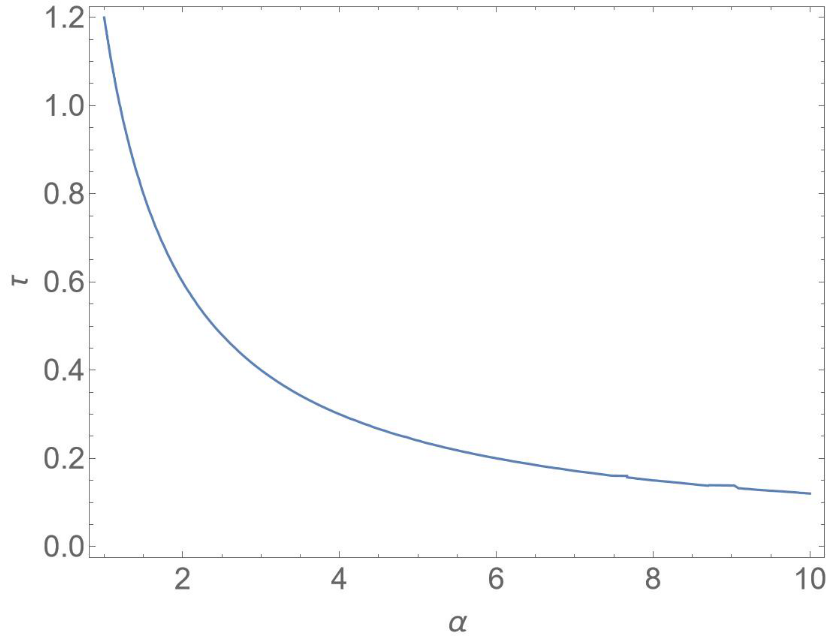

Fig.1: Relationship between dimensionless parameters τ and α.

Now the displacement reads

$$u(r,\bar{t})=\frac{s_0\omega}{r^2\tau^3}\left(r\tau(c\bar{t}\tau+s_0)-(c\bar{t}\tau-r\tau+s_0)(\tau(c\bar{t}+r)+s_0)\mathrm{ArcCoth}\left(\frac{r\tau}{c\bar{t}\tau+s_0}\right)\right). \quad (23)$$

Boundary condition

The boundary condition at the rapidly evolving surface supplies information for the determination of the parameter τ:

$$\left(1+\nu(\tau^2-2)\right)\mathrm{ArcCoth}(\tau)+\frac{(\nu-1)\beta}{2s_0c^2\rho\omega}\tau^3+(2\nu-1)\tau=0. \quad (24)$$

Due to the branch cut discontinuity of the ArcCoth function, and in order to maintain the meaningfulness of Eq. (24), τ is only admissible in the interval $(1,1.996784)$.

Finally, the triple of equations (23), (24) and compatibility condition (21) constitute the solution to the explosion problem in the region $r>s(\bar{t})=\tau c\bar{t}+s_0$.

Numerical Illustration

For illustration purposes, we consider an isotropic linear elastic infinite media with Poisson ratio $\nu=0.3$, $\lambda=\frac{3}{2}\mu$, $\mu=0.2$, $s_0=5$, $\beta=1$, $\rho=1500$, all in appropriate units. In this instance, it is calculated from the above relations that $\omega=-0.354452\beta$, and $\tau=1.5$ approximately. The resulting displacement field, spherical surface evolution, and pressure within the cavity are given by

$$\begin{cases} u(r,\overline{t}) = -\dfrac{1.16529}{r^2}\left(\begin{array}{l} 1.15r(0.0248428\overline{t}+5)-(1.15(r+0.0216025\overline{t})+5) \\ \times(-1.15r+0.0248428\overline{t}+5)\text{ArcCoth}\left(\dfrac{1.15r}{0.0248428\overline{t}+5}\right) \end{array}\right) \\ \mathrm{P}(t) = \left(0.0248428\overline{t}+5\right)^{-1} \\ s(\overline{t}) = 0.0248428\overline{t}+5. \end{cases} \tag{25}$$

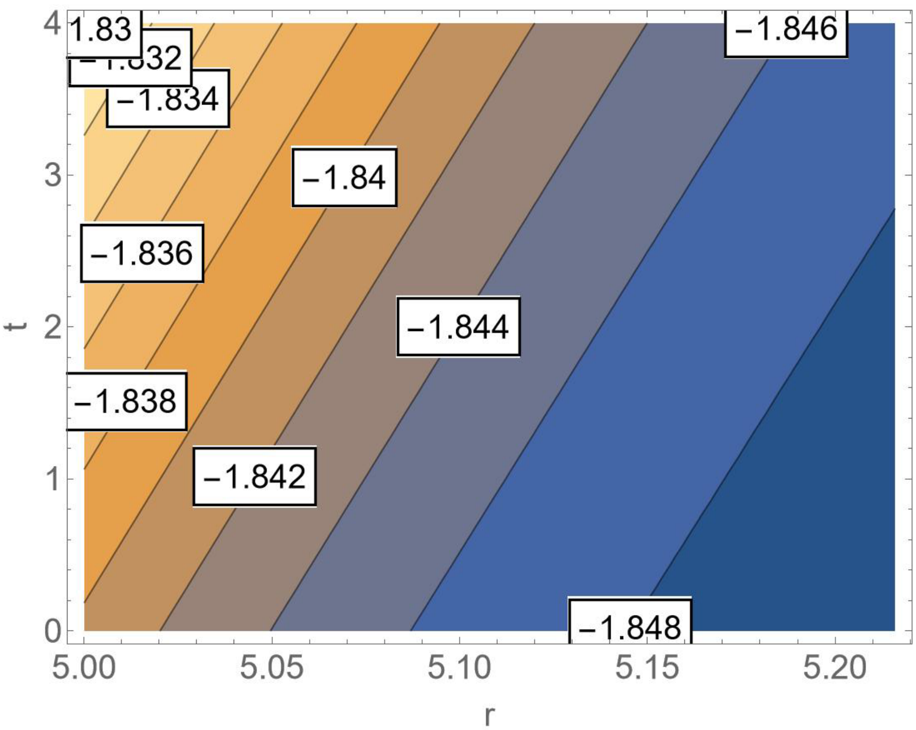

Fig. 2: Illustrative displacement field evolution in the region ahead of a rapidly expanding spherical cavity as given by Eq. (25)

The rapidly evolving surface and the ensuing compaction are demonstrated in the contour plot given in Fig.2 above. It can be deduced from Fig.2 that the spherical cavity's surface evolves linearly in the time, while lesser absolute values of the displacement $u(r,t)$ as progression is made into the far field, per time, depict decreasing compression.

References

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