Materials Research Proceedings 31 (2023) 336-346 https://doi.org/10.21741/9781644902592-35

Finite electroelastic deformation of dielectric semilinear hyperelastic tubes

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Keywords: Dielectric, Finite Deformation, Electroelasticity, Cylindrical Tube, Hyperelasticity

Abstract. This study examines the finite electroelastic deformation problem of extension and inflation coupling of dielectric semilinear hyperelastic tubes with closed ends under the influence of internal pressure, axial loads and radial electric field. The laws of thermodynamics and Coleman-Noll procedure are used to derive the electroelastic constitutive model of the tube. The solution of the consequent electromechanical field equations shows that the applied radial electric field associated with the equal and opposite charges on the electrode coated surfaces contributes to both internal pressure and axial loads of the closed tube. Furthermore, it is obtained that the stress propagation in dielectric semilinear hyperelastic solids is sensitive to the electric displacement field generated within the solids while the accompanying electric field interacts with the deformation of the solids. Finally, and among other things, the graphical illustration shows that the radial electric field generated within the tube increases with the increasing azimuthal stretch.

Introduction

Dielectric elastomers (DEs) are smart materials that change their mechanical behaviour in response to the application of electric field. These materials belong to a class of electroactive polymers, which generate finite deformation under the action of external electric field [1-9]. The high elastic strain energy density, electromechanical coupling and fast actuation speeds make DEs extremely attractive for widespread applications. For instance, the out-of-plane deformation response of an edge-clamped DE membrane has been investigated as a potentials replacement for the passive diaphragm in a left ventricular assist device; and using a DE membrane eliminates the need for a separate actuation source in the prosthetic pump, this enables a simpler, lighter, and more compact device that mimics the behaviour of the natural heart [2]. Other applications of DEs include but not limited to soft robotics, actuators, sensors, pvalves. energy harvesting, adaptive optics and haptic feedback.

In respect of the widespread applications of dielectric elastomers, and since the pioneering work of Toupin [10], the studies of electroelastic deformation of these materials have continued to attract considerable interest of reserachs in the fields of applied mathematics, physics, chemistry, and engineering. Melnikov and Ogden [11] used the reduced energy procedure and solved the problem of extension and inflation of a circular cylindrical incompresible tube made of dielectric elastomer based on neo-Hookean, Ogden and Gent material models; Saxena et al. [12] obtained the solution of combined extension, torsion and inflation in compressible electroelastomeric thin tubes; Dorfmann and Ogden [13] analysed the influence of deformation dependent permitivity on the elastic response of a finitely deformed dielectric incompressible thick-walled tube made of electro-sensitive neo-Hookean and Gent materials; Fu et al. [14] derived a reduced electroelastic plate model which describes the incremental formulation of an electrodes-coated dielectric plate that takes the leading-order thickness effect into account; and Abd-alla et al. [15] formulated and solved a mathematical model for longitudinal wave propagation in a magnetoelastic hollow circular cylinder of anisotropic material under the influence of initial hydrostatic stress to mention a few. The present study incorporates material incompressibility constraint into the rencent work of Fadodun et al. [1] and solve the problem of extension-inflation coupling of dielectric semilinear hyperelastic tube subject to internal pressure, axial load, and a radial electric field. Electroelasticty field equations, together with the consequent constitutive model and accompanying boundary conditions are used to obtain the solution of the problem. The rest of the paper is as follow: scetion two highlights the notaion, section three gives the basic equations of electroelasticity, section four details the application of electroelasticity theory presented in the previous sections to the problem of extension and inflation coupling of a thick circular cylindrical tube, section five gives the solution of the resulting electroelastic field equations, while section six concludes the study.

Notation

In this study, we employ upper-case boldface such as **A**, **B** for second-rank tensors. The tensors $\boldsymbol{A^T}$ and A^{-1} denote the transpose and inverse of tensor $\boldsymbol{A}$ respectively. Both upper- and lower-cases with overhead arrow such as $\vec{A}$ and $\vec{n}$ denote vectors, and parameters such as $p, \eta, F, F_{\text{red}}$ denote scalars.

Kinematics

Let a stress-free dielectric semilinear hyperelastic solid occupying the reference configuration $\Omega_0 \subset R^3$,with smooth boundary $\partial\Omega_0$ deform onto current/deformed configuration $\boldsymbol{\Omega}$, with smooth boundary $\partial\Omega$, when subject to electric field and/ or mechanical loads. Let $\vec{N}$ and $\vec{n}$ denote the outward unit normal vectors on boundaries $\partial\Omega_0$ and $\partial\Omega$ respectively, and let the vector function $\vec{\varphi}$

$$\vec{\varphi}: \overline{\Omega_0} \rightarrow \overline{\Omega}, \tag{1}$$

give the deformation of the body such that $\vec{x} = \vec{\varphi}(\vec{X})$, where $\vec{X}$ is the position vector of an arbitrary point in Ω_0 prior to the deformation and $\vec{x}$ is the position vector of the corresponding point in $\boldsymbol{\Omega}$. The closures $\overline{\Omega_0}$ and $\overline{\Omega}$ in Eq. (1) are defined by

$$\overline{\Omega_0} = \Omega_0 \cup \partial\Omega_0 \quad \text{and} \quad \overline{\Omega} = \Omega \cup \partial\Omega.$$

The deformation gradient **F** is defined by

$$\boldsymbol{F} = \text{Grad}\,\vec{x} = \text{Grad}\,\vec{\varphi}(\vec{X}), \tag{2}$$

where Grad is the gradient operator with respect to Ω_0. At an arbitrary point $\vec{X}$, the determinant of deformation gradient $\det(\boldsymbol{F}) > 0$ measures the local volume change. The left polar decomposition of **F** gives

$$\boldsymbol{F} = \boldsymbol{O}^D\,\boldsymbol{U}, \tag{3}$$

where $\boldsymbol{U}$ is the right stretch symmetric tensor and $\boldsymbol{O}^D$ is the orthogonal rotation tensor. The tensors $\boldsymbol{U}$ and $\boldsymbol{O}^D$ are obtained from the relations

$$\boldsymbol{U} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}} = \sqrt{\boldsymbol{C}} \quad and \quad \boldsymbol{O}^D = \boldsymbol{F}\,\boldsymbol{U}^{-1} \tag{4}$$

where $\boldsymbol{F}^T$ is the transpose of $\mathbf{F}$, $\boldsymbol{U}^{-1}$ is the inverse of $\boldsymbol{U}$ and $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy-Green deformation tensor [1].

Eulerian Electroelasticity Field Equations

The problem of electroelasticity being one of electromechanical problems, the basic equations are those of electrostatic and elasticity. Let $\boldsymbol{T}$ denote the total stress tensor and suppose $\vec{E}$ and $\vec{\mathrm{D}}$ denote the Eulerian electric field and electric displacement vectors respectively. For a purely electrostatic situation in the absence of mechanical body force, magnetic field, free electric current and free volumetric charges, the electroelasticity field equations are

$$\operatorname{div}\boldsymbol{T} = \vec{0},\ \ \operatorname{curl}\vec{E} = \vec{0}\ \ \text{ and }\ \operatorname{div}\vec{D} = 0, \tag{5}$$

where div and curl are the divergence and rotor operators with respect to the deformed configuration Ω. Note that the electric body force has been incorporated through $\boldsymbol{T}$, which by virtue of angular momentum balance is symmetric.

Let $\overrightarrow{E^*}$ and $\overrightarrow{D^*} = \epsilon_0\overrightarrow{E^*}$ denote the corresponding fields in the free space outside the body, where ϵ_0 is the permitivity of free space, both $\overrightarrow{E^*}$ and $\overrightarrow{D^*}$ satisfy Eqs. 5_2 and 5_3 respectively. The standard boundary conditions accompanying Eq. (5) are

$$\boldsymbol{T}\,\vec{n} = \overrightarrow{t_a} + \overrightarrow{t_m^*},\qquad \vec{n}\times\left(\overrightarrow{E^*} - \vec{E}\right) = \vec{0},\quad \vec{n}\cdot\left(\overrightarrow{D^*} - \vec{D}\right) = \sigma_f \quad \text{on} \quad \partial\Omega \tag{6}$$

where σ_f is the free surface charge on $\partial\Omega$ per unit area, $\overrightarrow{t_a}$ is the surface mechanical traction on $\partial\Omega$ per unit area and $\overrightarrow{t_m^*} = \boldsymbol{T}^*\,\vec{n}$ is the load due to the Maxwell stress [1, 11]

$$\boldsymbol{T}^* = \epsilon_0\,\overrightarrow{E^*}\otimes\overrightarrow{E^*} - \tfrac{1}{2}\epsilon_0\left(\overrightarrow{E^*}\cdot\overrightarrow{E^*}\right)\mathbf{I}\,. \tag{7}$$

Lagrangian Electroelasticity Field Equations

Let $\boldsymbol{P}$ denote the total first Piola-Kirchhoff stress tensor and suppose $\overrightarrow{E_L}$ and $\overrightarrow{D_L}$ denote the Lagrangian electric field and electric displacement vectors respectively. The electromechanical field variables $\boldsymbol{P}, \overrightarrow{E_L}, \overrightarrow{D_L}$ and $\boldsymbol{T}, \vec{E}, \vec{D}$ are related by

$$\boldsymbol{P} = \det(\boldsymbol{F})\,\boldsymbol{F}^{-1}\boldsymbol{T}\,,\ \ \overrightarrow{E_L} = \boldsymbol{F}^T\vec{E},\ \ \overrightarrow{D_L} = \det(\boldsymbol{F})\,\boldsymbol{F}^{-1}\vec{D}, \tag{8}$$

where the tensors $\boldsymbol{F}^{-1}$ is the inverses of the tensors $\boldsymbol{F}$.

The corresponding electroelasticity field equations in Lagrangian forms are

$$\operatorname{Div}\boldsymbol{P} = \vec{0},\ \ \operatorname{Curl}\overrightarrow{E_L} = \vec{0},\ \ \operatorname{Div}\overrightarrow{D_L} = 0 \tag{9}$$

where Div and Curl are the divergence and rotor operators with respect to the reference configuration Ω_0.

Similarly, the standard boundary conditions associated with Eq. 9 are

$$\boldsymbol{P}\vec{N} = \overrightarrow{t_F} + \overrightarrow{t_{mF}^*},\ \ \vec{N}\times\left(\boldsymbol{F}^T\overrightarrow{E^*} - \overrightarrow{E_L}\right) = \vec{0},\ \ \vec{N}\cdot\left(\det(\boldsymbol{F})\boldsymbol{F}^{-1}\overrightarrow{D^*} - \overrightarrow{D_L}\right) = \sigma_F \text{ on } \ \partial\Omega_0 \tag{10}$$

where σ_F is the free surface charge on $\partial\Omega_0$ per unit area, $\overrightarrow{t_F}$ is the surface mechanical traction on Ω_0 per unit area, $\overrightarrow{t_{mF}^*} = \boldsymbol{P}^* \vec{N}$ and

$$\boldsymbol{P}^* = \det(\boldsymbol{F})\, \boldsymbol{F}^{-1} \boldsymbol{T}^*,$$

is the pull back version of the Maxwell stress T^* [1, 11].

Electroelastic Constitutive Model for Incompressible Dielectric Hyperelastic Solids

In view of Fadodun et al. [1], Mehnikov and Ogden [11] and Dorfmann and Ogden [13], let $\Phi^*(\boldsymbol{F}, \overrightarrow{D_L})$ denote the electroelastic Helmholtz's free-energy function per unit volume, where $\boldsymbol{F}$ and $\overrightarrow{D_L}$ are the independent variables. Using the Coleman-Noll procedure, the total first Piola-Kirchhoff stress tensor **P** and Lagrangian electric field vector $\overrightarrow{E_L}$ are obtained by

$$\boldsymbol{P} = \frac{\partial \Phi^*(F, \overrightarrow{D_L})}{\partial \boldsymbol{F}} \text{ and } \overrightarrow{E_L} = \frac{\partial \Phi^*(\boldsymbol{F}, \overrightarrow{D_L})}{\partial \overrightarrow{D_L}}. \tag{11}$$

Let $\Phi(F)$ denote the strain energy per unit volume for purely elastic materials. In the case of an isotropic semilinear hyperelastic solids, the function $\Phi(F)$ reads

$$\Phi(F) = \mu_e I_{\mathbb{1}}^{\mathbb{2}}(\boldsymbol{U} - \boldsymbol{I_0}) + \frac{1}{2}\lambda_e I_{\mathbb{1}}(\boldsymbol{U} - \boldsymbol{I}_0)^2, \tag{12}$$

where $I_{\mathbb{1}}(\boldsymbol{U} - \boldsymbol{I}_0)$ is the first invariant of the tensor $(\boldsymbol{U} - \boldsymbol{I}_0)$, $\boldsymbol{I}_0$ is the unit tensor in the reference configuration Ω_0 and λ_e, μ_e are the material Lame constants [1, 16, 17].

On the basis of free-energy function in Eq. 12, and using the first and second laws of thermodynamics, Fadodun et al. [1] proposed electroelastic Helmholtz's free energy function $\Phi^*(\boldsymbol{F}, \overrightarrow{D_L})$

$$\Phi^*(\boldsymbol{F}, \overrightarrow{D_L}) = \mu_e I_{\mathbb{1}}^{\mathbb{2}}(\boldsymbol{U} - \boldsymbol{I}_0) + \frac{1}{2}\lambda_e I_{\mathbb{1}}(\boldsymbol{U} - \boldsymbol{I}_0)^2 + \frac{1}{2\epsilon}\overrightarrow{D_L} \cdot \boldsymbol{U} \cdot \overrightarrow{D_L}, \tag{13}$$

for the dielectric semilinear hyperelastic solid under consideration, where ϵ is the electric permitivity of the material.

Substituting Eq. 13 into Eq. 11 gives the electro-sensitive stress tensor

$$\boldsymbol{P} = \frac{\partial \Phi^*(\boldsymbol{F}, \overrightarrow{D_L})}{\partial \boldsymbol{F}} = 2\mu_e \boldsymbol{F}^T + \left((\lambda_e I_{\mathbb{1}}(\boldsymbol{U} - \boldsymbol{I}_0) - 2\mu_e)\boldsymbol{I}_0 + \frac{1}{2\epsilon}\left(\overrightarrow{D_L} \otimes \overrightarrow{D_L}\right)\right)\boldsymbol{O}^{DT}, \tag{14}$$

and deformation-dependent electric field vector

$$\overrightarrow{E_L} = \frac{\partial \Phi^*(\boldsymbol{F}, \overrightarrow{D_L})}{\partial \overrightarrow{D_L}} = \frac{1}{\epsilon}\boldsymbol{U}\overrightarrow{D_L}, \tag{15}$$

as the electroelastic constitutive model for the solids, where $\otimes$ is the tensor product [1].

In the case of an incompressible dielectric semilinear hyperelastic material, the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ in Eq. 14 assumes the form

$$\boldsymbol{P} = \frac{\partial \Phi^*(\boldsymbol{F}, \overrightarrow{D_L})}{\partial \boldsymbol{F}} = 2\mu_e \boldsymbol{F}^T + \left((\lambda_e I_{\mathbb{1}}(\boldsymbol{U} - \boldsymbol{I}_0) - 2\mu_e)\boldsymbol{I}_0 + \frac{1}{2\epsilon}\left(\overrightarrow{D_L} \otimes \overrightarrow{D_L}\right)\right)\boldsymbol{O}^{DT} - \eta \boldsymbol{F}^{-T}, \tag{16}$$

where η is the Lagrange multiplier associated with the incompressibility constraint $\det(\mathbf{F}) = 1$.

Application: Extension and Inflation Coupling of a Thick Electroelastic Tube

The theory of electroelasticity summarized in the previous sections is now specialized for application to the problem of combined extension and inflation of a relatively thick-walled circular cylindrical tube. The tube under consideration has closed ends and is subject to internal pressure, axial load and a radial electric field generated by a potential difference between flexible electrodes coated on its inner and outer radial surfaces.

Let $\vec{R}(R,\Theta,Z)$ be the position vector of an arbitrary point in the electroelastic tube Ω_0 prior to the deformation and let $\vec{r}(r,\theta,z)$ denote the position vector of the correspooding point in the deformed tube Ω. The reference configuration Ω_0 of the tube is described by

$$A \le R \le B, \qquad 0 \le \Theta \le 2\pi, \qquad 0 \le Z \le L, \tag{17}$$

where A and B are the inner and outer radii and L is the length of the tube in Ω_0.
Invoking the constraint of circular symmetric, the deformed configuration Ω is defined by

$$a \le r \le b, \qquad 0 \le \theta \le 2\pi, \qquad 0 \le z \le l, \tag{18}$$

where a and b are the inner and outer radii and l is the length of the tube in Ω. Note that (R,Θ,Z) and (r,θ,z) are the cylindrical polar coordinates in the reference configuration Ω_0 and deformed configuration Ω respectively.
Using the incompressibility constraint of the tube material, the deformation is defined by the relations [11]

$$r = r(R) = \sqrt{a^2 + \lambda_z^{-1}(R^2 - A^2)}, \qquad \theta = \Theta, \qquad z = \lambda_z Z, \tag{19}$$

where λ_z is the uniform axial stretch of the tube. In view of Eq. (2), the deformation gradient $\boldsymbol{F}$ is

$$\boldsymbol{F} = \mathrm{Grad}\,\vec{r} = \frac{\partial r}{\partial R}\,\overrightarrow{e_r^*} \otimes \overrightarrow{E_R^*} + \frac{r}{R}\,\overrightarrow{e_\theta^*} \otimes \overrightarrow{E_\Theta^*} + \frac{\partial z}{\partial Z}\,\overrightarrow{e_z^*} \otimes \overrightarrow{E_Z^*}, \tag{20}$$

where $\overrightarrow{e_r^*}, \overrightarrow{e_\theta^*}, \overrightarrow{e_z^*}$ and $\overrightarrow{E_R^*}, \overrightarrow{E_\Theta^*}, \overrightarrow{E_Z^*}$ are the orthonormal basis vectors in Ω and Ω_0 respectively.
Substituting Eq. 19 into Eq. 20 and introducing $\lambda_\theta = \frac{r}{R}$ as the azimuthal stretch give

$$\boldsymbol{F} = \frac{1}{\lambda_z\lambda_\theta}\,\overrightarrow{e_r^*} \otimes \overrightarrow{E_R^*} + \lambda_\theta\,\overrightarrow{e_\theta^*} \otimes \overrightarrow{E_\Theta^*} + \lambda_z\,\overrightarrow{e_z^*} \otimes \overrightarrow{E_Z^*}. \tag{21}$$

Using Eqs. 3, 4 and 21 gives

$$\boldsymbol{U} = \boldsymbol{F} \quad \text{and} \quad \boldsymbol{O}^D = \boldsymbol{I}_0. \tag{22}$$

Let the parameters λ_a and λ_b be defined by

$$\lambda_a = \frac{a}{A}, \quad \lambda_b = \frac{b}{B}, \quad \text{and} \quad b = r(B). \tag{23}$$

Using Eq. 19_1 and Eq. 23 give the relations

$$\frac{\lambda_a^2\lambda_z-1}{\lambda_\theta^2\lambda_z-1}=\frac{B^2}{A^2}\left(\frac{\lambda_b^2\lambda_z-1}{\lambda_\theta^2\lambda_z-1}\right)=\frac{R^2}{A^2}. \tag{24}$$

Equation 24 shows that the term $R^2\left(\lambda_\theta^2\lambda_z-1\right)$ defined by

$$R^2\left(\lambda_\theta^2\lambda_z-1\right)=B^2(\lambda_b^2\lambda_z-1)=A^2(\lambda_a^2\lambda_z-1)=\tau^2,\quad \tau\in R \tag{25}$$

is independent of both R and r; and for the case of inflation of cylindrical tube at fixed length $\lambda_\theta^2\lambda_z-1>0$ for $\lambda_b\leq\lambda_\theta\leq\lambda_a$.

Components of Stress Tensor and Electric Field Vector

In view of Melnikov and Ogden [11], an applied potential difference between the electrode coated surfaces $R = A$ and $R = B$ generates a radial electric field within the tube and is associated with equal and opposite charges on the electrode coated surfaces. By Gauss's law and neglecting the edge effect, the electric field vector vanishes outside the tube. The components of electric displacement vector $\overrightarrow{D_L}$ within the tube assume the form

$$D_R=D(R)\neq 0,\ D_\Theta=0,\ \ D_Z=0, \tag{26}$$

where D_R, D_Θ and D_Z are the radial, azimuthal and axial components of $\overrightarrow{D_L}$ respectively and $D(R)$ is a function of radius R only.

Using Eqs. 15, 21, 22 and 26 gives

$$E_R=\frac{1}{\lambda_\theta\lambda_z\epsilon}D_R,\qquad E_\Theta=0,\qquad E_Z=0, \tag{27}$$

where E_R, E_Θ and E_Z are the radial, azimuthal and axial components of $\overrightarrow{E_L}$ respectively.

Substituting Eqs. 21, 22, and 26 into Eq. 16 gives

$$P_{RR}=\frac{2\mu_e}{\lambda_\theta\lambda_z}+\lambda_e\left(\frac{1}{\lambda_\theta\lambda_z}+\lambda_\theta+\lambda_z-3\right)-2\mu_e+\frac{1}{2\epsilon}D_R^2-\eta\lambda_\theta\lambda_z,\quad P_{R\Theta}=0,\quad P_{RZ}=0, \tag{28}$$

$$P_{\Theta\Theta}=2\mu_e\lambda_\theta+\lambda_e\left(\frac{1}{\lambda_\theta\lambda_z}+\lambda_\theta+\lambda_z-3\right)-2\mu_e-\frac{\eta}{\lambda_\theta},\quad P_{\Theta R}=0,\quad P_{\Theta Z}=0, \tag{29}$$

$$P_{ZZ}=2\mu_e\lambda_z+\lambda_e\left(\frac{1}{\lambda_\theta\lambda_z}+\lambda_\theta+\lambda_z-3\right)-2\mu_e-\frac{\eta}{\lambda_z},\quad P_{ZR}=0,\quad P_{Z\Theta}=0, \tag{30}$$

where P_{ij}, $i,j=R,\Theta,Z$ are the components of total first Piola-Kirchhoff stress tensor **P**.

Solution of Reduced Electroelasticity Field Equations for Extension-Inflation Coupling of Dielectric Semilinear Hyperelastic Tube

In view of Eqs. 28-30, Eq. 9_1 reduces to

$$\frac{d}{dR}P_{RR}+\frac{1}{R}(P_{RR}-P_{\Theta\Theta})=0. \tag{31}$$

Substituting Eqs. 28_1 and 29_1 into Eq. 31 gives

$$R\frac{d\eta}{dR} = -\left(\frac{2\mu_e+\lambda_e}{\lambda_\theta^4\lambda_z^3}\right)\left(\lambda_\theta^2\lambda_z - 1\right)^2 + \frac{1}{\epsilon\lambda_\theta\lambda_z}\left(\frac{1}{2}D_R^2 + RD_R\frac{dD_R}{dR}\right). \quad (32)$$

The form of electric field vector $\overrightarrow{E_L}$ in Eq. 27 shows that Eq. 9_2 is satisfied identically; and substituting Eq. 26 into Eq. 9_3 gives

$$\frac{d}{dR}(RD_R) = \frac{d}{dR}\left(RD(R)\right) = 0. \quad (33)$$

The integration of Eq. 33 gives

$$RD(R) = const. \quad (34)$$

At the coated surfaces $R = A$ and $R = B$, we have

$$R\,D(R) = A\,D(A) = B\,D(B). \quad (35)$$

Let the total charge at the coated surface $R = A$ be Q, then the total charge at the coated surface $R = B$ is -Q. The free surface charge densities per unit area are

$$\sigma_F = \frac{Q}{2\pi AL} \text{ on } R = A \text{ and } \sigma_F = -\frac{Q}{2\pi BL} \text{ on } R = B. \quad (36)$$

In view of the boundary condition Eq. 10_3 and setting $\overrightarrow{D^*} = \vec{0}$ yield

$$D(A) = \frac{Q}{2\pi AL} \quad \text{and} \quad D(B) = \frac{Q}{2\pi BL}. \quad (37)$$

Using Eqs. 35 and 37 gives

$$D_R = D(R) = \frac{Q}{2\pi RL}. \quad (38)$$

The combination of Eq. 38 and Eq. 27_1 yields [1]

$$E_R = \frac{1}{\lambda_\theta\lambda_z}\left(\frac{Q}{2\epsilon\pi RL}\right). \quad (39)$$

Substituting Eq. 38 into Eq.32 gives

$$d\eta = -\left(\frac{2\mu_e+\lambda_e}{\lambda_\theta^4\lambda_z^3}\right)\left(\lambda_\theta^2\lambda_z - 1\right)^2\frac{dR}{R} - \frac{1}{2\epsilon\lambda_\theta\lambda_z}\left(\frac{Q}{2\pi L}\right)^2\frac{1}{R^2}\frac{dR}{R}. \quad (40)$$

The relation $\lambda_\theta = \frac{r}{R}$ and knowing that $\frac{dr}{dR} = (\lambda_\theta\lambda_z)^{-1}$ gives

$$\frac{dR}{R} = \left(\frac{\lambda_\theta\lambda_z}{1-\lambda_\theta^2\lambda_z}\right)d\lambda_\theta. \quad (41)$$

Substituting Eq. 41 into Eq. 40 gives

$$d\eta = \left(\frac{2\mu_e+\lambda_e}{\lambda_\theta^3\lambda_z^2}\right)\left(\lambda_\theta^2\lambda_z - 1\right)d\lambda_\theta + \frac{1}{2\epsilon}\left(\frac{Q}{2\pi L}\right)^2 \frac{d\lambda_\theta}{R^2\left(\lambda_\theta^2\lambda_z-1\right)}. \tag{42}$$

Substituting Eq. 25 into Eq. 42 gives

$$d\eta = \left(\frac{2\mu_e+\lambda_e}{\lambda_\theta^3\lambda_z^2}\right)\left(\lambda_\theta^2\lambda_z - 1\right)d\lambda_\theta + \frac{1}{2\epsilon}\left(\frac{Q}{2\pi\tau L}\right)^2 d\lambda_\theta. \tag{43}$$

The integration of Eq. 43 gives

$$\eta = (2\mu_e + \lambda_e)\left(\frac{1}{\lambda_z}\ln\lambda_\theta + \frac{1}{2\lambda_z^2\lambda_\theta^2}\right) + \frac{1}{2\epsilon}\left(\frac{Q}{2\pi\tau L}\right)^2 \lambda_\theta + C, \tag{44}$$

where C is the constant of integration.
Substituting Eq. 44 into Eq. 28_1 gives

$$P_{RR} = (2\mu_e + \lambda_e)\left(\frac{1}{2\lambda_\theta\lambda_z} - \lambda_\theta\ln\lambda_\theta\right) + \lambda(\lambda_\theta + \lambda_z)$$

$$+\frac{1}{2\epsilon}\left(\frac{Q}{2\pi L}\right)^2\left(\frac{1}{R^2} - \frac{\lambda_\theta^2\lambda_z}{\tau^2}\right) - (2\mu_e + 3\lambda_e) - \lambda_\theta\lambda_z C. \tag{45}$$

The mechanical boundary conditions on the surfaces $R = A$ and $R = B$ are

$$P_{RR} = -p \quad \text{on} \quad R = A \quad \text{and} \quad P_{RR} = 0 \quad \text{on} \quad R = B, \tag{46}$$

where p is the internal pressure.
Substituting Eq. 45 into Eq. 46_2 gives the integration constant C

$$C = \left(\frac{2\mu_e+\lambda_e}{\lambda_b\lambda_z}\right)\left(\frac{1}{2\lambda_b\lambda_z} - \lambda_b\ln\lambda_b\right) + \frac{1}{2\epsilon\lambda_b\lambda_z}\left(\frac{Q}{2\pi L}\right)^2\left(\frac{1}{B^2} - \frac{\lambda_b^2\lambda_z}{\tau^2}\right) + \frac{\lambda(\lambda_b+\lambda_z)}{\lambda_b\lambda_z} - \left(\frac{2\mu_e+3\lambda_e}{\lambda_b\lambda_z}\right). \tag{47}$$

Substituting Eq. 47 into Eq. 45 gives

$$P_{RR} = (2\mu_e + \lambda_e)\left(\frac{1}{2\lambda_\theta\lambda_z} - \lambda_\theta\ln\lambda_\theta + \frac{\lambda_\theta}{\lambda_b}\left(\lambda_b\ln\lambda_b - \frac{1}{2\lambda_b\lambda_z}\right)\right) + (2\mu_e + 3\lambda_e)\left(\frac{\lambda_\theta}{\lambda_b} - 1\right)$$

$$\frac{1}{2\epsilon}\left(\frac{Q}{2\pi L}\right)^2\left(\frac{1}{R^2} - \frac{\lambda_\theta^2\lambda_z}{\tau^2} + \left(\frac{\lambda_b^2\lambda_z}{\tau^2} - \frac{1}{B^2}\right)\frac{\lambda_\theta}{\lambda_b}\right) + \lambda\left((\lambda_\theta + \lambda_z) - (\lambda_b + \lambda_z)\frac{\lambda_\theta}{\lambda_b}\right). \tag{48}$$

Substituting Eq. 48 into Eq. 46_1 gives the expression for the internal pressure p

$$p = (2\mu_e + \lambda_e)\left(\frac{1}{2\lambda_z}\left(\frac{\lambda_a}{\lambda_b^2} - \frac{1}{\lambda_a}\right) + \lambda_a\ln\left(\frac{\lambda_a}{\lambda_b}\right)\right) + \frac{1}{2\epsilon}\left(\frac{Q}{2\pi L}\right)^2\left(\frac{\lambda_a\lambda_z}{\tau^2}(\lambda_a - \lambda_b) - \frac{1}{A^2} + \frac{1}{B^2}\frac{\lambda_a}{\lambda_b}\right)$$

$$+\lambda_e\lambda_z\left(\frac{\lambda_a}{\lambda_b} - 1\right) + (2\mu_e + 3\lambda_e)\left(1 - \frac{\lambda_a}{\lambda_b}\right). \tag{49}$$

In addition to the internal pressure obtained in Eq. 49, we proceed to find the axial load F applied to the ends of the closed tube. The applied axial load F is defined by

$$F = 2\pi \int_A^B P_{ZZ}\ R dR. \tag{50}$$

Let the stresses $P_{\Theta^* R^*}$ and $P_{Z^* R^*}$ be defined by

$$P_{\Theta^* R^*} = P_{\Theta\Theta} - P_{RR} \quad \text{and} \quad P_{Z^* R^*} = P_{ZZ} - P_{RR}. \tag{51}$$

The axial stress P_{ZZ} is related to the radial stress P_{RR} and the stresses $P_{\Theta^* R^*}, P_{Z^* R^*}$ by

$$P_{ZZ} = \frac{1}{2}\left(\frac{1}{R}\frac{d}{dR}(R^2 P_{RR})\right) - \frac{1}{2}P_{\Theta^* R^*} + P_{Z^* R^*}. \tag{52}$$

Using Eqs. 52 and 46 in Eq. 50 gives

$$F = \pi A^2 p + \pi \int_A^B (2P_{Z^* R^*} - P_{\Theta^* R^*}) R\, dR. \tag{53}$$

It is obvious from Eq. 53 that the internal pressure *p* contributes to the axial load *F*; consequently the reduced axial load F_{red} is defined by

$$F_{red} = F - \pi A^2 p = \pi \int_A^B (2P_{Z^* R^*} - P_{\Theta^* R^*}) R\, dR. \tag{54}$$

Using Eqs. 28, 29 and 51 in Eq. 54 gives

$$F_{red} = \frac{(2\mu_e + \lambda_e)\tau^2}{\lambda_z} \ln\left(\frac{\lambda_a}{\lambda_b}\right) + (6\mu_e + \lambda_e)\tau^2 \left(\frac{\lambda_a}{\lambda_a^2\lambda_z - 1} - \frac{\lambda_b}{\lambda_b^2\lambda_z - 1}\right)$$

$$- \frac{1}{2\epsilon\sqrt{\lambda_z}}\left(\frac{Q}{2\pi L}\right)^2 \left(\tanh^{-1}(\sqrt{\lambda_z}\lambda_a) - \tanh^{-1}(\sqrt{\lambda_z}\lambda_b)\right)$$

$$- \left(\frac{4\mu_e + \lambda_e}{\lambda_z} + \lambda_e\lambda_z(\lambda_z - 1)\right)\tau^2 \left(\frac{1}{\lambda_a^2\lambda_z - 1} - \frac{1}{\lambda_b^2\lambda_z - 1}\right) + 2\mu\lambda_z\tau^2 \left(\frac{\lambda_a + \lambda_z - 3}{\lambda_a^2\lambda_z - 1} - \frac{\lambda_b + \lambda_z - 3}{\lambda_b^2\lambda_z - 1}\right). \tag{55}$$

Remark: It is obvious from Eqs. 49 and 55 that both internal pressure and axial load interact with the applied radial electric field associated with the equal and opposite charges on the electrode coated surfaces of the tube. These results are in agreement with study in literature.

Finally, in view of Eq. 39, the electric field generated within tube in now expressed in terms of azimutal stretch λ_θ

$$E_R = \frac{1}{\epsilon A \lambda_\theta \lambda_z}\left(\frac{Q}{2\pi L}\right)\sqrt{\frac{\lambda_\theta^2\lambda_z - 1}{\lambda_a^2\lambda_z - 1}}. \tag{56}$$

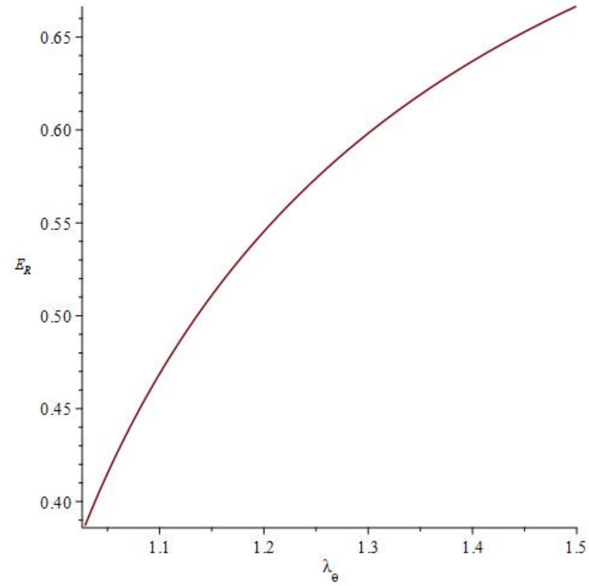

Fig.: This plot shows that the radial electric field generated within the tube increases with the increasing azimuthal stretch of the tube

Conclusion

The study formulated electroelastic constitutive model for incompressible dielectric semmilinear hyperelastic materials. The consequent electroelastic equations are used to solve the problem of extension and inflation coupling of circular cylindrical tube. It is obtained that the applied radial electric field associated with the equal and opposite charges on the electrode coated surfaces contributes to both internal pressure and axial loads of the cylindrical tube. The aplications of the results in this study include but not limited to actutors, sensors, and soft robotic.

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