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The dynamics of circular arches with multiple damage

CANNIZZARO Francesco[1,a *], FIORE Ilaria[1,b], GRECO Annalisa[1,c], CADDEMI Salvatore[1,d] and CALIO' Ivo[1,e]

[1] Department of Civil Engineering and Architecture, University of Catania, via Santa Sofia 64, 95123 Catania, Italy

[a] francesco.cannizzaro@unict.it, [b] ilaria.fiore@phd.unict.it, [c] annalisa.greco@unict.it, [d] salvatore.caddemi@unict.it, [e] ivo.calio@unict.it

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Abstract. The governing equation of the dynamics of the planar inextensible Euler-Bernoulli arch with multiple damage is tackled in this study by employing the distributional approach. Precisely, the presence of impairments is modelled via cracks that can be effectively embedded in the governing equation by means of the Dirac's delta generalised functions. The governing equation is defined over a unique integration domain. The proposed integration strategy leads to closed form expressions of the displacement mode shapes maintaining the size of the problem as that of the undamaged arch regardless the number of cracks located along the span. The latter advantage avoids the enforcement of continuity conditions at the discontinuous sections. The proposed solution extends the integration procedure proposed in the static context to the vibration analysis and allows determining the modal characteristics of damaged circular arches. The versatility of the obtained closed form solution allows a straightforward execution of parametric analyses and is here adopted to evaluate the sensitivity of the eigenproperties of the multi-cracked circular arch to the change of meaningful geometric and mechanical parameters.

Introduction

The static and dynamic behaviour of the arches was studied by the researchers both in the linear and nonlinear context by means of different approaches over the years, such as limit analysis, Finite Element analysis and the Discrete Macro-Element Model. Arches may present circular shape [1] or a variable curvature [2, 3], and are often studied considering only the flexural deformability, neglecting both shear and axial effects. With regard to damaged arches, the presence of impairments is usually tackled with concentrated models [4]. The concentrated damage can be modelled according to various strategies, and in this paper the well-known equivalent spring model [5] is adopted. This leads to the insertion of a rotational spring with constant stiffness in each cracked section, allowing dealing with linear governing equations.

Many studies analysed the in-plane or out-of-plane vibrations of damaged circular arches [6, 7, 8] . The presence of concentrated cracks along the axis of straight or curved beams requires the subdivision of the element into uniform sub-elements connected by a rotational spring. Therefore, it is necessary to enforce continuity conditions or, alternatively, the assemblage of finite elements, thus increasing the computational burden. Alternatively, several authors proposed new approaches which require solely the enforcement of the external boundary conditions irrespectively of the number of internal cracks. These solutions are available for both the static and dynamic behaviour of multi-cracked straight beams [9, 10], whilst solutions for curved beams are limited to the static problem only [11].

In this paper, a free vibration six order differential governing equation of a multi-cracked circular inextensible Euler-Bernoulli arch is presented, and the associated closed form solution of the vibration modes is derived by means of the application of the Laplace transform. It is worth highlighting that the equation of motion is defined over a unique integration domain, even if in presence of multiple cracks, and the evaluation of the natural frequencies requires the enforcement of six boundary conditions only, as in the continuous arch case. The proposed solution is numerically validated by comparing the natural frequencies and the mode shapes of a multi-cracked circular arch with the results obtained by means of a finite element model in which the cracks are modeled by means of equivalent internal rotational springs. Some parametric studies referred to multi-cracked Euler arches with variable intensity or position of the concentrated crack are also presented and discussed.

Closed-form integration of the free vibration equation of the multi-cracked circular arch

The differential equation governing the in plane free vibrations of a Euler-Bernoulli circular arch with radius r in presence of n concentrated cracks, is presented in this section, and the relevant closed form solution in terms of mode shapes is obtained.

The integration domain is represented by $\vartheta \in [\vartheta_{o1}, \vartheta_{o2}]$, being ϑ a zenithal angular coordinate, and $\vartheta_{o1}, \vartheta_{o2}$ the left and right end cross sections, respectively. The kinematics of the arch involves the following components depending on the angular coordinate ϑ and the time t: the radial $u_r(\vartheta,t)$ and tangential $u_t(\vartheta,t)$ displacements, as well as the rotation of the centroidal axis $\varphi(\vartheta,t)$. For the case of inextensible Euler-Bernoulli arch model, the following kinematic constraints among the bending curvature $\chi(\vartheta,t)$, $\varphi(\vartheta,t)$, $u_r(\vartheta,t)$ and $u_t(\vartheta,t)$ hold:

$$\chi(\vartheta,t) = \frac{1}{r}\varphi^{I}(\vartheta,t) \quad , \quad u_t^{I}(\vartheta,t) = u_r(\vartheta,t) \quad , \quad \varphi(\vartheta,t) = -\frac{1}{r}\left[u_t(\vartheta,t) + u_t^{II}(\vartheta,t)\right] \tag{1}$$

where the Roman numbers superscripts indicate derivatives with respect to ϑ. Considering an Euler-Bernoulli circular arch characterized by n double-edge concentrated cracks, assumed to be always open, at cross sections ϑ_i, $i = 1,...,n$, its free vibration motion is ruled by the following sixth order governing equation, expressed in terms of tangential displacement $u_t(\vartheta,t)$:

$$\left[\frac{\partial^6}{\partial\vartheta^6} + 2\frac{\partial^4}{\partial\vartheta^4} + \frac{\partial^2}{\partial\vartheta^2} + \frac{mr^4}{EJ_r}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial\vartheta^2} - 1\right)\right]u_t(\vartheta,t) + r\sum_{i=1}^{n}\Delta\varphi_i(t)\left[\frac{\partial}{\partial\vartheta} + \frac{\partial^3}{\partial\vartheta^3}\right]\delta(\vartheta - \vartheta_i) = 0 \tag{2}$$

where the rotary inertia has been neglected in the dynamic equilibrium and denoting with m the distributed mass. The presence of n concentrated cracks implies rotation discontinuities $\Delta\varphi_i(t)$ at the cracked cross-sections ϑ_i, $i = 1,...,n$. These singularities are taken into account in the summation in Eq. 2, characterized by sequences of n occurrences of first and third derivatives of Dirac's deltas. This strategy allows preserving the definition of the governing differential equation over the entire domain $\vartheta \in [\vartheta_{o1}, \vartheta_{o2}]$ as already proposed in the static case [11]. The governing free vibration equation provided by Eq. 2 can be reformulated as follows:

$$\left[\frac{\partial^6}{\partial\vartheta^6} + 2\frac{\partial^4}{\partial\vartheta^4} + \frac{\partial^2}{\partial\vartheta^2} + \mu^4\left(1 - \frac{\partial^2}{\partial\vartheta^2}\right)\right]\phi_t(\vartheta) = \sum_{i=1}^{n}\lambda_i\left[\phi_t^{I}(\vartheta_i^-) + \phi_t^{III}(\vartheta_i^-)\right]\left(\frac{\partial}{\partial\vartheta} + \frac{\partial^3}{\partial\vartheta^3}\right)\delta(\vartheta - \vartheta_i) \tag{3}$$

where the frequency parameter $\mu^4 = \omega^2 mr^4 / EJ_r$ is introduced and the tangential displacement $u_t(\vartheta,t) = \phi_t(\vartheta)\sin\omega t$ is expressed as the product of two functions, namely $\phi_t(\vartheta)$, depending solely on the non-dimensional angular coordinate ϑ, and a harmonic function $\sin\omega t$, being ω the radial frequency. Analogously, the unknown rotation discontinuities $\Delta\varphi_i(t) = \Delta\phi_{\varphi,i}\sin\omega t$, appearing in Eq. 2, can be expressed in terms of the tangential displacement mode shape $\phi_t(\vartheta)$ as follows:

$$\Delta\phi_{\varphi,i} = -\frac{\lambda_i}{r}\left[\phi_t^I\left(\vartheta_i^-\right) + \phi_t^{III}\left(\vartheta_i^-\right)\right] \tag{4}$$

where $\lambda_i = EJ_r / r K_i^{eq}$ represents the dimensionless crack compliance related to the stiffness of the equivalent rotational spring.

The closed form solution of Eq. 3 can be retrieved by means of the application of Laplace transform, leading to the expression of the tangential displacement mode shape as follows:

$$\phi_t(\vartheta) = \sum_{k=1}^{6} C_k \, h_k(\vartheta) + \sum_{i=1}^{n} \lambda_i \left[\phi_t^I\left(\vartheta_i^-\right) + \phi_t^{III}\left(\vartheta_i^-\right)\right] \overline{h}_i(\vartheta) \tag{5}$$

where:

$$\begin{aligned} & h_k(\vartheta) = e^{\alpha_k \vartheta}, \quad k = 1,\ldots,6; \\ & \overline{h}_i(\vartheta) = \left[\sum_{k=1}^{6} \frac{\alpha_k^3 + \alpha_k}{6\alpha_k^5 + 8\alpha_k^3 + 2(1-\mu^4)\alpha_k} e^{\alpha_k(\vartheta - \vartheta_i)}\right] U(\vartheta - \vartheta_i), \quad i = 1,\ldots,n \end{aligned} \tag{6}$$

$$\begin{aligned} C_k = \frac{1}{6\alpha_k^5 + 8\alpha_k^3 + 2(1-\mu^4)\alpha_k} \Big[& \left(\alpha_k^5 + 2\alpha_k^3 + (1-\mu^4)\alpha_k\right)\phi_t(0) + \left(\alpha_k^4 + 2\alpha_k^2 + 1 - \mu^4\right)\phi_t^I(0) + \\ & + \left(\alpha_k^3 + 2\alpha_k\right)\phi_t^{II}(0) + \left(\alpha_k^2 + 2\right)\phi_t^{III}(0) + \alpha_k \phi_t^{IV}(0) + \phi_t^V(0) \Big] \end{aligned} \tag{7}$$

being $\alpha_k, k = 1,\ldots,6$ the roots of the sixth order polynomial $p(s;\mu^4) = s^6 + 2s^4 + \left(1-\mu^4\right)s^2 + \mu^4$ that can be inferred in closed form.

The solution, as it stands in Eq. 5, does not provide an explicit expression for the tangential displacement mode shape $\phi_t(\vartheta)$ since it depends on the values of its first and third distributional derivatives $\phi_t^I(\vartheta)$ and $\phi_t^{III}(\vartheta)$, evaluated at ϑ_i^-. The latter can be expressed in explicit form and substituted into Eq. 5, providing the following explicit expression of the tangential displacement mode shape function:

$$\phi_t(\vartheta) = \sum_{k=1}^{6} C_k f_k(\vartheta) \tag{8}$$

where the functions $f_k(\vartheta)$, $k = 1,\ldots,6$, are defined by the following expressions:

$$f_k(\vartheta) = h_k(\vartheta) + \sum_{i=1}^{n} \lambda_i \left[g_{I,k}(\vartheta_i^-) + g_{III,k}(\vartheta_i^-)\right] \overline{h}_i(\vartheta) \tag{9}$$

and the terms $g_{D,k}(\vartheta_i^-)$, $D = 0, I, \ldots, V$, $k = 1,\ldots,6$, are defined by the following expressions:

$$g_{D,k}(\vartheta_i^-) = h_k^{(D)}(\vartheta_i^-) + \sum_{m=1}^{i-1} \lambda_m \left[g_{I,k}(\vartheta_m^-) + g_{III,k}(\vartheta_m^-)\right] \overline{h}_m^{(D)}(\vartheta_i^-). \tag{10}$$

The radial displacement and rotation mode shape functions can be easily obtained, in view of the solution provided in Eq. 8, by means of the kinematic constraints expressed by Eq. 1. Each of the two end cross-sections provides three boundary conditions depending on the kinematic and mechanical conditions of the restraint. It has to be pointed out that the proper integration constants are the values of the derivatives of ϕ_t at $\vartheta = 0$ of the arch denoted as $\phi_t^{(D)}(0)$, D indicating the order of the derivative $D = 0, I, \ldots, V$, appearing in Eq. 7. The latter are to be determined according to specified boundary conditions dependent on the external constraints acting on the circular arch getting rid of the dependency of the frequency parameter μ^4 appearing in Eq. 7.

Numerical applications

Double cracked arch (FEM comparison)

In this sub-section an application regarding a clamped-clamped double cracked full circular arch (the angular span of the arch is 180°) is reported, aiming at validating the proposed closed-form solution. The radius of the axis of the arch is r = 2 m, the cross section is rectangular with a base b = 40 mm and a height h = 50 mm. The mass density is equal to $\rho = 7850$ kg/m^3, whilst the Young's modulus is E = 210000 MPa. The two cracks are localised in two sections placed at $\vartheta_1 = -\pi / 4$ and $\vartheta_2 = \pi / 6$ with respect to the axis of symmetry of the arch, and with intensity parameters $\lambda_1 = 0.0289$ and $\lambda_2 = 0.0833$, respectively . The intensity parameters are related to crack depth/cross section height ratios $\beta_1 = 0.3$, $\beta_2 = 0.5$, according to the model proposed in [12]. The analytical natural frequencies and modes of vibration have been obtained by means of the proposed model and compared to the results obtained by means of a FEM model implemented in the software environment SAP2000 in accordance with the assumptions of infinite axial and shear stiffness and where the cracks have been modeled by considering an equivalent rotational spring for each cracked cross section. The first three natural frequencies for the damaged configuration of the arch are shown in Table 1, whilst the first three displacement mode shapes for the proposed (dashed black line) and the FEM (solid grey line) models have been reported in Fig. 1.

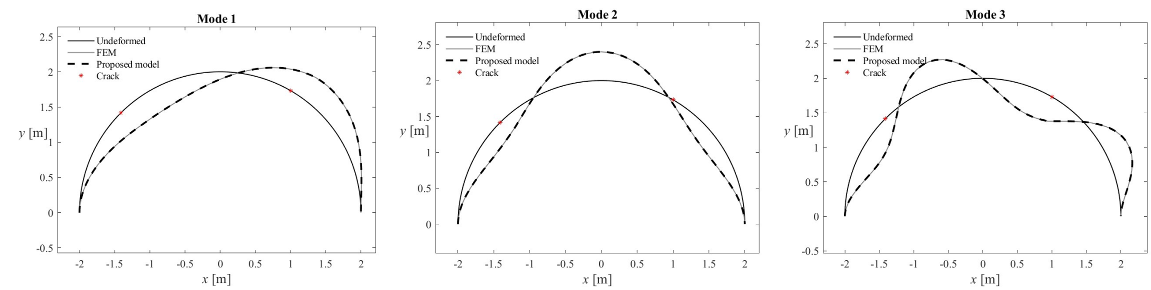

Figure 1 - First three mode shapes of the double cracked circular arch

Table 1 - Modal frequencies comparison

	Mode 1	Mode 2	Mode 3
Proposed model	12.7331 Hz	28.4654 Hz	52.4530 Hz
FEM model	12.7372 Hz	28.4786 Hz	52.4725 Hz

The agreement between the proposed and the FEM models, both in terms of frequency and mode shapes, is excellent. The subtle differences might be due to the discretized approach of the FEM model compared to the proposed continuous approach.

Parametric study

In this sub-section a parametric study is presented considering a clamped-clamped full circular arch with three cracks (at the two ends and at the midspan sections) with increasing intensities. The crack intensity parameters λ assume for all the cracks the following values: 0, 0.2, 0.4, 0.6, 0.8, 1, 2, 3, 4, 6, 8, 10. Furthermore, in view of the parametric nature of the study the cross section of the arch has not been specified, hence the crack intensity parameters can be related to specific crack depths in accordance with the case study at hand.

In Fig. 2 the variation of the frequency parameter μ^4 against the crack intensity parameter λ, for the first two vibration modes, is reported. In the same Figure, the frequency parameters μ^4 for a clamped-clamped and a three-hinged full circular arch are also reported for comparison.

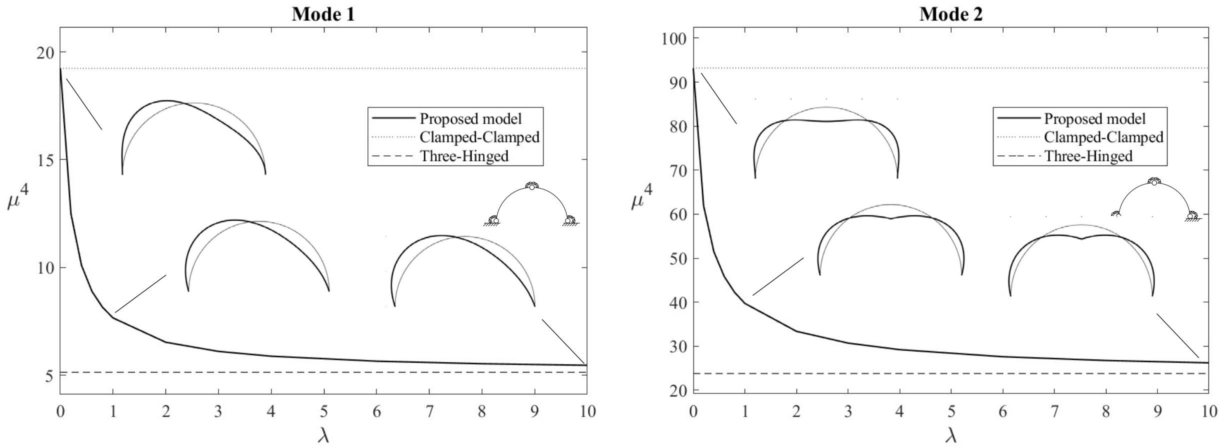

Figure 2 - Frequency parameter against the crack intensity parameter for the first and second mode of vibration

The results obtained in terms of frequency parameter show a monotonic trend as the value of intensity parameter increases, as expected. Furthermore, it can be observed that the frequency parameter is equal to the value of the clamped-clamped arch for $\lambda = 0$ and tends to the value of the three-hinged arch for higher values of the intensity parameter.

Conclusions

This work presented a free vibration six order differential governing equation of a multi-cracked circular inextensible Euler-Bernoulli arch and the associated closed form solution in terms of mode shapes. The proposed governing equation is defined over a unique integration domain regardless of the number of cracks, and the evaluation of the natural frequencies requires the enforcement of six boundary conditions only at the two ends of the arch, as in the continuous arch case, avoiding the introduction of continuity conditions at the cracked sections. The proposed procedure was validated by comparison with numerical solutions obtained with a classic FEM approach and the closed form solution was used for a representative parametric study.

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