Theoretical and Applied Mechanics - AIMETA 2022 Materials Research Forum LLC
Materials Research Proceedings **26** (2023) 381-385 https://doi.org/10.21741/9781644902431-62

Materials with memory: some new results in viscoelastic models

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Keywords: Materials with Memory, Viscoelasticity, Magneto-Viscoelasticity, Aging

Abstract. The classical model of viscoelastic body is reconsidered. As well known the deformation response of a material is termed viscoelastic when it does not depend only on the status of the material in the considered time, but also on its deformation history. Furthermore, the system is dissipative: such dissipative nature induce the use of viscoelastic materials in devising anti-seismic dissipators. Aiming to model new and innovative materials different forms of the relaxation modulus which characterises the response of the material are considered. Cases of a non-classical relaxation modulus are studied. Thus, a relaxation modulus which may be unbounded or less regular or modified to describe a material whose mechanical response is changed when the material with time, this phenomenon is usually termed "aging" are investigated. Finally, the viscoelastic response can be controlled on devising magneto-viscoelastic materials via injection of micro or nano particles magnetically sensible.

Introduction

The model of viscoelastic body, according to Fabrizio and Morro [18, 2] are the background for the present investigation. The viscoelastic body is assumed homogeneous and isotropic so that the dependence on the spatial variable can be omitted. Conversely, the dependence with respect to time is not only via the present time, but also through the whole *deformation history* of the material. Accordingly, the quantities of interest are:

$\mathbf{E} = \mathbf{E}(t)$ *strain tensor*

$\mathbf{T} = \mathbf{T}(t)$ *stress tensor*

$\mathbf{G} = \mathbf{G}(t)$ *relaxation modulus*

$\mathbf{G}_0 = \mathbf{G}(0)$ *initial relaxation modulus*

wherein no space dependence is indicated; these quantities are connected via the *constitutive assumptions*

$$\mathbf{T}\,(t) = \int_0^\infty \mathbf{G}(\tau)\ \dot{\mathbf{E}}\,(t-\tau)\ d\tau \quad , \quad \mathbf{G}(t) = \mathbf{G}_0 + \int_0^t \dot{\mathbf{G}}(s)\ ds \qquad (1)$$

or equivalently, when $\mathrm{E}_t(\tau)$ denotes the **strain past history**

$$\mathbf{T}\,(t) = \mathbf{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbf{G}}(\tau)\ \mathbf{E}^t\,(\tau)\ d\tau \quad , \quad \mathbf{E}^t(\tau) := \mathbf{E}(t-\tau) \qquad (2)$$

Classical problem: regular kernel
The previous assumptions imply that, in the case the relaxation functions satisfies all the written regularity requirements, the problem, termed *classical*, in the one-dimensional case can be written as

$$u_{tt} = G(0)u_{xx} + \int_0^t \dot{G}(t-\tau)u_{xx}(\tau)d\tau + f \qquad (3)$$

where, respectively, *u* and *f* denote the displacement and the external force which takes into account also the history of the material and, hence, is not zero. In addition, in the one-dimansional case, the tensor G is represented by a real valued function. The problem is assigned prescribing the following initial and boundary conditions:

$$u(\cdot,0) = u_0,\ \ u_t(\cdot,0) = u_1 \text{ in } \Omega\,; \quad u = 0 \text{ on } \Sigma = \partial\Omega \times (0,T)\ . \qquad (4)$$

Then, the existence and uniqueness result by Dafermos [17] can be applied. The *relaxation modulus* is the quantity, in this model, whose properties describe the behaviour of the material under investigation. The classical requirements on the relaxation modulus, the kernel in the integro-differential equation (3) are

$$\dot{G} \in L^1(\mathbb{R}^+)\ \ ,\ \ G(t) = G_0 + \int_0^t \dot{G}(s)\ ds\ ,\ G(\infty) := \lim_{t\to\infty} G(t)\ ,\ G_0 := \lim_{t\to 0} G(t) \qquad (5)$$

Hence, *G* enjoys the *fading memory property* that is

$$\forall\epsilon > 0\,\exists\,\tilde{a} = a\left(\epsilon, E^t\right) \in \mathbb{R}^+ s.t.\ \forall a > \tilde{a}, \left|\int_0^\infty \dot{G}(s+a)E^t(s)\ ds\right| < \epsilon \qquad (6)$$

whose physical meaning is that the effects very far in the past are negligible. In the following a list of different generalisations are given: they are devised to describe different materials for which the classical assumptions on the relaxation modulus cannot be adopted. The assumptions (5), which correspond to G continuous and differentiable positive valued with a negative derivative for all positive times, approaching to zero as t goes to infinity. In the case G twice differentiable thermodynamical compatibility implies also that

$$\dot{G} < 0, \ddot{G} > 0, \forall t \in (0.T), \forall T > 0$$

Non classical problems: Singular Kernel and "aging"
When we consider the case of a relaxation modulus $\lim_{t\to 0} G(t) = +\infty$. In this case [11, 4] as well the 3-dimensional generalisation [6], the problem cannot be formulated via (3) since both G(0) is not defined and $\dot{G} \notin L^1$, hence a different approach is adopted. Specifically, approximated problems are introduced and, via a suitable limit procedure, existence and uniqueness of the solution of an initial boundary value problem is proved [11].

On the other hand, another generalisation which is suggested by applications consists in taking into account the fact that, in general, the response of the material changes over time due to the natural deterioration of the material itself. These effects are studied in [15, 16] an overview is given in [5].

Magneto-viscoelastic materials
The model we adopted to describe the interaction between the viscoelastic body and an external magnetic field goes back to in [19] later revisited in [1], coupling between viscoelasticity and magnetisation.

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Regular kernel

The regular kernel problem, which corresponds to consider the viscoelastic solid characterised by a relaxation function which satisfies the classical assumptions is studied in [12], in the one dimensional case and in [14] in the three dimensional case. The one-dimensional problem reads

$$\begin{cases} u_{tt} - G(0)u_{xx} - \displaystyle\int_0^t G'(t-\tau)u_{xx}(\tau)d\tau - \frac{\lambda}{2}(\Lambda(\mathbf{m})\cdot\mathbf{m})_x = f\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega\times(0,T) \\ \mathbf{m}_t + \mathbf{m}\dfrac{|\mathbf{m}|^2-1}{\varepsilon} + \lambda\Lambda(\mathbf{m})u_x - \mathbf{m}_{xx} = 0\,, \end{cases} \tag{7}$$

where $\mathbf{\Omega} = (0, 1)$ and $\mathbf{m} = (0, m_1, m_2)$, denotes the magnetization vector, orthogonal to the conductor, since $\mathbf{u} = (u, 0, 0)$, when both quantities are written in $\mathbb{R}^3$. In addition, Λ is a linear operator defined by $\Lambda(\mathbf{m}) = (0, m_2, m_1)$, the scalar function u is the displacement in the direction of the conductor itself, here identified with the x−axis and λ is a positive parameter. In addition, the term f represents an external force which also includes the deformation history up to t = 0. Letting ν be the outer unit normal at $\partial\mathbf{\Omega}$, initial and boundary conditions are given as follows

$$u(\cdot,0) = u_0 = 0,\ \ \mathbf{m}(\cdot,0) = \mathbf{m}_0, \quad |\mathbf{m}_0| = 1 \quad \text{in } \Omega\,,\ \ , \tag{8}$$

$$u = 0, \qquad \frac{\partial\mathbf{m}}{\partial\boldsymbol{\nu}} = 0 \qquad \text{on} \quad \Sigma = \partial\Omega\times(0,T)\,, \tag{9}$$

Under the assumptions

$G(t)$ X $C^2(0, T)$, u_0 X $H^1_0(\Omega)$, u_1 X $L^2(\Omega)$, m_0 X $H^1(\Omega)$ and f X $L^2(\Omega\times(0, T))$,

the existence and uniqueness of the solution to the problem given by (7)-(8), is proved in [12]. The corresponding 3-dimensional problem is studied in [14].

Non classical problems: Singular Kernel and "aging"

The same problem, when the relaxation function is assumed to be unbounded at the initial time t = 0, that is coupling, now, the singular viscoelastic behaviour with the magnetisation effects is studied in [9] wherein a singular viscoelastic behaviour is coupled with the magnetic filed. Again, the introduction of suitable approximated problems and a limit procedure, allow to prove [9] existence of the solution of an initial boundary value problem.

A different generalisation is considered in [10] wherein the one-dimensional viscoelasticity model is modified to take into account the so called aging effects. The term aging is adopted to indicate that the response of the material is not unchanged with time. That is, if a long term use of the material is considered, then the response of the viscoelastic material is, in general, different from the initial one, after a long time. To model this behaviour the relaxation function is assumed to depend on the two time variables t and τ not only through their difference as in the classical model, see formulae (1) and (2) but we consider G a function of the two different time variables t and τ here the two time variables are independent. The regular magneto-viscoelasticity problem with aging in [10] is proved to admit an unique solution.

Conclusions

The wide variety of applications of viscoelastic materials for instance in the area of the study of attenuators which are devised to possibly prevent damages via dissipation of extra energy in the case of seismic events as mentioned in [3].

On the other hand, the more and more widely spread use of magnetically sensible particles for instance in gels which can be modelled as viscoelastic materials induces to further investigate this subject, see, for instance, [21], [22] and [23] to have an idea of the different kinds of applications which go from rheology to biomedical applications.

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