Some remarks on the evaluation of work and dissipated energy associated with rate-independent hysteretic forces

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Abstract. We provide closed-form expressions to compute the path-dependent work performed by generalized rate-independent hysteretic forces, simulated by using a brand-new model denominated Vaiana-Rosati Model (VRM). In particular, such expressions are valid over a generic generalized displacement interval. Furthermore, we provide a closed-form expression for evaluating the dissipated energy associated with the work done by a rate-independent hysteretic force when a full cycle of periodic generalized displacement is applied.

Introduction

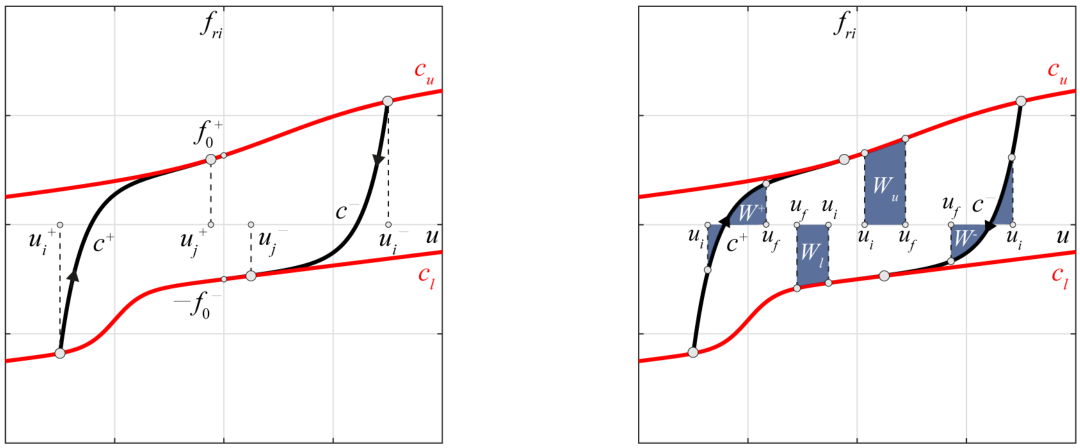
In structural dynamics, the problem of evaluating the amount of mechanical energy dissipated by rate-independent hysteretic mechanical systems and materials plays a crucial role since such an issue plays a paramount role in vibrations control. In particular, with reference to the seismic protection of structures and objects contained in them [1], the evaluation of the dissipated energy is necessary for the modeling and design of isolation bearings [2-4] and energy dissipation devices [5].

The complexity of the nonlinear response exhibited by rate-independent hysteretic mechanical systems strongly influences the shape of generalized force-displacement hysteresis loops, which can be *symmetric* or *asymmetric*. Over the years, many researchers have proposed phenomenological models for evaluating the generalized force by means of different types of equations.

As regards the simulation of *symmetric* hysteresis loops, the Bouc-Wen model [6-7] and its subsequent extensions [8] are the most employed differential models. On the other hand, non-differential models, such as algebraic or transcendental ones, have been recently formulated by Vaiana et al. [9]. As far as the simulation of *asymmetric* hysteresis loops is concerned, it is possible to employ modified versions of the Bouc-Wen model [10] or, alternatively, recently developed models belonging to the generalized class proposed by Vaiana et al. [11].

In general, hysteresis phenomena are so complex that it is quite complicated to describe them by means of a single model. Conversely, Vaiana and Rosati [12] have recently proposed a preliminary formulation of a novel rate-independent hysteretic model capable of simulating a wide range of hysteresis loops in a unified manner. This model provides a closed-form expression to evaluate the output variable, thus allowing for significant advantages in terms of computational efficiency [13-14].

In particular, in this paper, after a short description of the Vaiana-Rosati model, we illustrate the closed-form expressions that have been derived to compute the incremental generalized work performed by an hysteretic force; on the basis of this result, we derive the closed-form expression for the evaluation of the dissipated energy associated with a full hysteresis loop.



(a) (b)

***Figure 1**: VRM formulation: curves c^+, c^-, c_u, c_l, (a) and incremental generalized work associated with each curve (b).*

Brief Review of the VRM

In this section, we briefly describe the Vaiana-Rosati Model (VRM). The model can accurately simulate the response of a great variety of rate-independent hysteretic mechanical systems and materials. In particular, it can reproduce the complex hysteretic behavior characterized by *asymmetric*, *pinched*, *S-shaped*, *and flag-shaped* hysteresis loops, or by a combination of them, providing *closed-form expressions* for the computation of the generalized rate-independent force. Moreover, the model allows for uncoupled modeling of the generic loading and unloading phases thanks to a set of parameters that can be simply calibrated due to their clear mechanical meaning and direct relationship with experimental loops.

Fig. 1a shows a typical hysteresis loop that the VRM can reproduce. The generalized rate-independent force f_{ri}, during a generic loading phase $\left(\dot{u} > 0\right)$, can be evaluated as:

$$f_{ri}\left(u,u_j^+\right)=\begin{cases} c^+\left(u,u_j^+\right) & \text{if} \quad u_i^+ \le u < u_j^+ \\ c_u\left(u\right) & \text{if} \quad u > u_j^+, \end{cases} \tag{1}$$

whereas, during the generic unloading one $\left(\dot{u} < 0\right)$, it can be computed as:

$$f_{ri}\left(u,u_j^-\right)=\begin{cases} c^-\left(u,u_j^-\right) & \text{if} \quad u_j^- < u \le u_i^- \\ c_l\left(u\right) & \text{if} \quad u < u_j^-, \end{cases} \tag{2}$$

In Eq. (1), c^+ and c_u describe, respectively, the generic loading curve and the upper limiting curve, which have the following expressions:

$$c^+\left(u,u_j^+\right)=\beta_1^+ e^{\beta_2^+ u} - \beta_1^+ + \frac{4\gamma_1^+}{1+e^{-\gamma_2^+\left(u-\gamma_3^+\right)}} - 2\gamma_1^+ + k_b^+ u + f_0^+ - \frac{1}{\alpha^+}\left[e^{-\alpha^+\left(+u-u_j^+ +\bar{u}^+\right)} - e^{-\alpha^+\bar{u}^+}\right], \tag{3}$$

$$c_u\left(u\right)=\beta_1^+ e^{\beta_2^+ u} - \beta_1^+ + \frac{4\gamma_1^+}{1+e^{-\gamma_2^+\left(u-\gamma_3^+\right)}} - 2\gamma_1^+ + k_b^+ u + f_0^+, \tag{4}$$

whereas, in Eq. (2), c^- and c_l define, respectively, the generic unloading curve and the lower limiting curve; their expressions read:

$$c^-\left(u,u_j^-\right)=\beta_1^- e^{\beta_2^- u} - \beta_1^- + \frac{4\gamma_1^-}{1+e^{-\gamma_2^-\left(u-\gamma_3^-\right)}} - 2\gamma_1^- + k_b^- u - f_0^- + \frac{1}{\alpha^-}\left[e^{-\alpha^-\left(-u+u_j^- +\bar{u}^-\right)} - e^{-\alpha^-\bar{u}^-}\right], \tag{5}$$

$$c_l\left(u\right)=\beta_1^- e^{\beta_2^+ u}-\beta_1^- +\frac{4\gamma_1^-}{1+e^{-\gamma_2^-\left(u-\gamma_3^-\right)}}-2\gamma_1^- +k_b^- u-f_0^- . \tag{6}$$

The parameters $k_b^+,\, f_0^+,\, \alpha^+,\beta_1^+,\, \beta_2^+,\, \gamma_1^+,\, \gamma_2^+,\, \gamma_3^+\, \left(k_b^-,\, f_0^-,\, \alpha^-,\beta_1^-,\, \beta_2^-,\, \gamma_1^-,\, \gamma_2^-,\, \gamma_3^-\right)$, that appear in the expressions of the generalized rate-independent hysteretic force, represent the set of eight parameters that control the generic loading (unloading) phase and need to be calibrated from experimental or numerical tests. Concerning the internal variable $u_j^+\left(u_j^-\right)$ characterizing the generic loading (unloading) phase, its expression is reported in Vaiana and Rosati [12], whereas $\overline{u}^+\left(\overline{u}^-\right)$ is an internal model parameter expressed as a function of $\alpha^+\left(\alpha^-\right)$.

Generalized Work and Dissipated Energy

In the sequel, we show that the closed-form expressions for the generalized rate-independent hysteretic force, described in the previous Section, allow us to derive closed-form expressions for the evaluation of the area under the four curves characterizing the VRM (Fig. 1b) as:

$$W_{ri}=\int_{u_i}^{u_f} f_{ri}\left(u\right)du. \tag{7}$$

In fact, the quantity in Eq. (7) represents the path-dependent work performed by the generalized force f_{ri} over a generic generalized displacement interval $\left[u_i, u_f\right]$, and will be referred to as generalized rate-independent hysteretic work W_{ri}. The path-dependent work will be first derived for the four curves defining the model; subsequently, the relevant expressions will be used to evaluate the closed-form expression of the energy dissipated in a full hysteresis loop.

Incremental Generalized Work

The closed-form expressions of the incremental generalized rate-independent hysteretic work associated with the four VRM curves are obtained by recalling Eqs. (1) and (2) and using them in Eq. (7). In this way, during the generic loading phase $\left(\dot{u}>0\right)$, the generalized work is:

$$W_{ri}=\begin{cases} W^+ & \text{if} \quad u_i^+ \le u < u_j^+ \\ W_u & \text{if} \quad\quad u > u_j^+, \end{cases} \tag{8}$$

whereas, in the generic unloading phase $\left(\dot{u}<0\right)$, it can be computed as:

$$W_{ri}=\begin{cases} W^- & \text{if} \quad u_j^- < u \le u_i^+ \\ W_l & \text{if} \quad\quad u < u_j^-, \end{cases} \tag{9}$$

where W_u $\left(W^+\right)$ represents, on the basis of Eq. (7), the area under the curve c_u $\left(c^+\right)$ over a generic generalized displacement interval $\left[u_i, u_f\right]$, whereas W_l $\left(W^-\right)$ is the area under the curve c_l $\left(c^-\right)$ over a generic generalized displacement interval $\left[u_f, u_i\right]$, as shown in Fig. 1b.

To evaluate the expressions of W_u, W^+, W_l, and W^-, it is more convenient to rewrite the expressions of the four functions defining the curves, as the sum of terms that depend, respectively, upon suitable subsets of the model parameters, namely:

$$c_u\left(u\right)=c_a^+\left(\beta_1^+,\beta_2^+\right)+c_b^+\left(\gamma_1^+,\gamma_2^+,\gamma_3^+\right)+c_c^+\left(k_b^+,f_0^+\right), \tag{10}$$

$$c^+\left(u,u_j^+\right)=c_a^+\left(\beta_1^+,\, \beta_2^+\right)+c_b^+\left(\gamma_1^+,\, \gamma_2^+,\gamma_3^+\right)+c_c^+\left(k_b^+,\, f_0^+\right)+c_d^+\left(\alpha^+\right), \tag{11}$$

$$c_l\left(u\right)=c_a^-\left(\beta_1^-,\beta_2^-\right)+c_b^-\left(\gamma_1^-,\gamma_2^-,\gamma_3^-\right)+c_c^-\left(k_b^-,f_0^-\right), \tag{12}$$

$$c^-\left(u,u_j^-\right)=c_a^-\left(\beta_1^-,\beta_2^-\right)+c_b^-\left(\gamma_1^-,\gamma_2^-,\gamma_3^-\right)+c_c^-\left(k_b^-,f_0^-\right)+c_d^-\left(\alpha^-\right). \tag{13}$$

Thus, the expressions of W_u and W^+ can be derived by integrating Eqs. (10)-(11) over $\left[u_i,u_f\right]$:

$$W_u=\int_{u_i}^{u_f} c_u\left(u\right)du=\int_{u_i}^{u_f}\left(c_a^++c_b^++c_c^+\right)du=W_a^++W_b^++W_c^+, \tag{15}$$

$$W^+=\int_{u_i}^{u_f} c^+\left(u,u_j^+\right)du=\int_{u_i}^{u_f}\left(c_a^++c_b^++c_c^++c_d^+\right)du=W_a^++W_b^++W_c^++W_d^+, \tag{14}$$

where the terms W_a^+, W_b^+, W_c^+, and W_d^+ are given by:

$$W_a^+=\frac{\beta_1^+}{\beta_2^+}\left(e^{\beta_2^+u_f}-e^{\beta_2^+u_i}\right)-\beta_1^+\left(u_f-u_i\right), \tag{18}$$

$$W_b^+=2\gamma_1^+\left\{\frac{2\ln\left[e^{-\gamma_2^+\left(u_f-\gamma_3^+\right)}+1\right]}{\gamma_2^+}-\frac{2\ln\left[e^{-\gamma_2^+\left(u_i-\gamma_3^+\right)}+1\right]}{\gamma_2^+}+\left(u_f-u_i\right)\right\}, \tag{19}$$

$$W_c^+=\frac{k_b^+}{2}\left(u_f^2-u_i^2\right)+f_0^+\left(u_f-u_i\right), \tag{20}$$

$$W_d^+=\frac{1}{\left(\alpha^+\right)^2}e^{\alpha^+\left(\bar{u}^+-u_j^+\right)}\left[\alpha^+\left(u_f-u_i\right)e^{-\alpha^+u_j^+}+e^{-\alpha^+u_f}-e^{-\alpha^+u_i}\right]. \tag{21}$$

It is important to note that Eq. (18) assumes finite values only if $\beta_2^+\neq 0$. Similarly, Eq. (19) is defined only if $\gamma_2^+\neq 0$. However, if $\beta_2^+=0$ it turns out to be $c_a^+=0$, and, consequently, $W_a^+=0$. Similarly, if $\gamma_2^+=0$ then $c_b^+=0$ and $W_b^+=0$. Eq. (21) does not present the same problem for $\alpha^+=0$ since, according to the conditions indicated by Vaiana and Rosati [12], this parameter needs to be always greater than zero. Similarly, the expressions of W_l and W^- are obtained by integrating Eqs. (12)-(13) over $\left[u_f,u_i\right]$; the relevant expressions W_a^-, W_b^-, W_c^-, and W_d^-, are omitted for brevity.

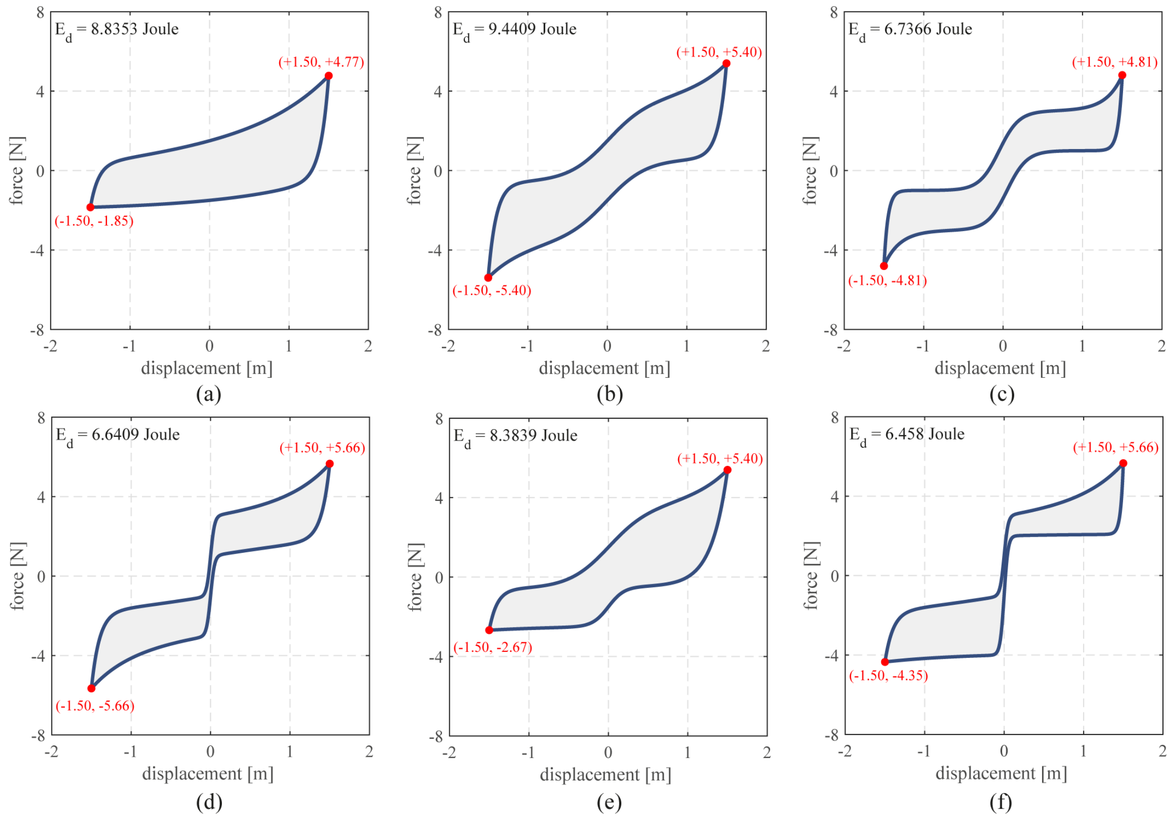


***Figure 2**: Dissipated energy associated with asymmetric (a), pinched (b), S-shaped (c), flag-shaped (d) hysteresis loops and two combinations of them (e)-(f), simulated by the VRM.*

Energy Dissipated in a Full Hysteresis Loop

Finally, the closed-form expression of the energy dissipated in a full hysteresis loop is obtained by summing the areas under the four curves (Fig. 1b). Recalling Eqs. (10)-(13), the energy dissipated in a full loop E_d can be obtained, by exploiting the additivity rule of definite integrals, as:

$$E_d = \int_{u_i^+}^{u_i^-} \left(c_a^+ + c_b^+ + c_c^+\right) du + \int_{u_i^+}^{u_j^+} c_d^+ \, du + \int_{u_i^-}^{u_i^+} \left(c_a^- + c_b^- + c_c^-\right) du + \int_{u_i^-}^{u_j^-} c_d^- \, du, \tag{22}$$

where all the quantities appearing in Eq. (22) can be calculated using the general expressions of the incremental work previously calculated, paying attention to set the proper displacement interval. Fig. 2 shows the values of the *dissipated energy* E_d, calculated using Eq. (22), for different hysteresis loops simulated by the VRM and obtained upon application of a full cycle of harmonic (sinusoidal) generalized displacement.

Conclusions

We have exploited one of the peculiar features of the VRM, that is, the analytical expressions assumed for the functions ruling the rate-independent hysteretic behavior, to derive closed-form formulas providing the path-dependent work done by an arbitrary rate-independent hysteretic force. Furthermore, analytical expressions of the energy dissipated in a full hysteresis loop have been explicitly computed. They represent the prerequisite to model the evolution of the hysteretic loop shape as a function of the energy dissipated in previous loading histories of cyclic nature, an issue that will be pursued in forthcoming papers.

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