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An unconditionally stable time integration for the dynamics of elastic beams and shells in finite motions

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Abstract. This work presents a numerical framework for long dynamic simulations of structures made of multiple thin shells undergoing large deformations. The C1-continuity requirement of the Kirchhoff-Love theory is met in the interior of patches by cubic NURBS approximation functions with membrane locking avoided by patch-wise reduced integration. A simple penalty approach for coupling adjacent patches, applicable also to non-smooth interfaces and non-matching discretization is adopted to impose translational and rotational continuity. A time-stepping scheme is proposed to achieve energy conservation and unconditional stability for general nonlinear strain measures and penalty coupling terms, like the nonlinear rotational one for thin shells. The method is a modified mid-point rule with the internal forces evaluated using the average value of the stress at the step end-points and an integral mean of the strain-displacement tangent operator over the step computed by time integration points.

Introduction

One-step implicit time integration methods such as Newmark's schemes are only conditionally stable when used in large deformation analyses [1]. Simo and Tarnow proposed a simple method that guarantees unconditional stability by conserving the algorithmic energy in elastodynamics [2]. However, energy conservation is lost for other structural models as the Kirchhoff-Love theory, more efficient in the terms of spatial DOFs for thin shell problems, where the strain-displacement relationship is no longer quadratic. This work presents a numerical framework for long term dynamic simulations of structures made of multiple thin shells undergoing large deformations. The C1-continuity requirement of the Kirchhoff-Love theory is met in the interior of patches by cubic NURBS approximation functions, according to the isogeometric concept, with membrane locking avoided by patch-wise reduced integration [4]. A simple penalty approach for coupling adjacent patches, applicable to either smooth or non- smooth interfaces and either matching or non-matching discretizations is adopted to impose translational and rotational continuity [5]. The time-stepping scheme of Simo and Tarnow is generalized to achieve energy conservation for generally nonlinear strain measures and penalty coupling terms, like the nonlinear rotational one for thin shells. The method is based on a particular integral mean of the internal forces over the step, that includes Simo and Tarnow's method as a reduced quadrature rule, and has unconditional stability.

Large deformation dynamic problem with general nonlinear strain measure

Let us consider a generic elastic body characterized by a linear elastic constitutive law

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$$

where $\boldsymbol{\sigma}$ is the vector collecting the stress/generalized stress components, $\boldsymbol{\varepsilon}$ is the vector collecting the strain/generalized strain components and C is the constitutive matrix. The strain is linked to the displacement field u by means of a differential operator generally nonlinear in u:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{1}$$

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Applying a spatial discretization technique, for instance the finite element method or the isogeometric analysis, the displacement is approximated at element level as

$$\boldsymbol{u} = \mathbf{N}_{\mathrm{u}}\mathbf{u}_{\mathrm{e}}$$

where matrix $\mathbf{N}_{\mathrm{u}}$ collect the spatial shape functions. The strain energy of the body can be expressed as a sum of element contributions as $\Phi \equiv \sum_e \Phi_{\mathrm{e}}(\mathbf{u}_{\mathrm{e}})$, with

$$\Phi_{\mathrm{e}}(\mathbf{u}_{\mathrm{e}}) \equiv \int_{\Omega_e} \frac{1}{2}\boldsymbol{\varepsilon}^{\boldsymbol{T}}\mathbf{C}\boldsymbol{\varepsilon}\, \mathrm{d}\Omega_{\mathrm{e}} \tag{2}$$

where Ω_{e} is the element domain. The first variation of the strains in Eq.(1) can be written as

$$\delta\boldsymbol{\varepsilon} = \mathbf{B}_{\mathrm{e}}\delta\mathbf{u}_{\mathrm{e}} \tag{3}$$

where matrix B is the strain-displacement tangent operator. The first variation of the element strain energy

$$\Phi'_{\mathrm{e}}(\mathbf{u}_{\mathrm{e}})\delta\mathbf{u}_{\mathrm{e}} \equiv \int_{\Omega_e} \delta\boldsymbol{\varepsilon}^{T}\mathbf{C}\boldsymbol{\varepsilon}\, \mathrm{d}\Omega_{\mathrm{e}} = \delta\mathbf{u}_{\mathrm{e}}^{\mathrm{T}} \int_{\Omega_e} \mathbf{B}_{\mathrm{e}}{}^{T}(\mathbf{u}_{\mathrm{e}})\boldsymbol{\sigma}(\mathbf{u}_{\mathrm{e}})\mathrm{d}\Omega_{\mathrm{e}} = \delta\mathbf{u}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{s}_e(\mathbf{u}_{\mathrm{e}}) \tag{4}$$

allows us to define the element internal force vector

$$\boldsymbol{s}_e(\mathbf{u}_{\mathrm{e}}) \equiv \int_{\Omega_e} \mathbf{B}_{\mathrm{e}}{}^{T}(\mathbf{u}_{\mathrm{e}})\boldsymbol{\sigma}(\mathbf{u}_{\mathrm{e}})\mathrm{d}\Omega_{\mathrm{e}} \quad \text{with} \quad \boldsymbol{\sigma}(\mathbf{u}_{\mathrm{e}}) = \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}_{\mathrm{e}}) \tag{5}$$

Similarly, the velocity field v is approximated consistently as

$$\boldsymbol{v} = \mathbf{N}_{\mathrm{u}}\mathbf{v}_{\mathrm{e}}$$

where $\mathbf{v}_{\mathrm{e}}$ are the discrete velocity DOFs. The kinetic energy is then sum of element contributions $\mathrm{T} \equiv \sum_e \mathrm{T}_{\mathrm{e}}(\mathbf{v}_{\mathrm{e}})$

$$\mathrm{T}_{\boldsymbol{e}}(\mathbf{v}_{\mathrm{e}}) \equiv \int_{\boldsymbol{\Omega_e}} \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\varepsilon}^{\boldsymbol{T}}\boldsymbol{\Xi}\boldsymbol{\varepsilon}\, \mathrm{d}\Omega_{\mathrm{e}} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{v}_{\boldsymbol{e}}^{\boldsymbol{T}}\boldsymbol{M}_{\boldsymbol{e}}\boldsymbol{v}_{\boldsymbol{e}} \tag{6}$$

where $\boldsymbol{\Xi}$ is a diagonal matrix with the mass density associated to each component of $\boldsymbol{v}$, which can be different for examples when v collects both translational and rotational velocities, and $\mathbf{M}_{\mathrm{e}}$ is the element mass matrix.

After a standard assemblage process, the semi-discrete equations of motion for the discretized body can be written, neglecting the damping, in terms of the global DOFs u and v as

$$\begin{cases} \mathbf{v} = \dot{\mathbf{u}} \\ \mathbf{M}\dot{\mathbf{v}} + \mathbf{s}(\mathbf{u}) = \mathbf{f} \end{cases} \tag{7}$$

where a dot denotes time derivative and $\mathbf{f}$ is the discrete load vector, while the total energy of the system is

$$\Pi(\mathbf{u}, \mathbf{v}) \equiv T(\mathbf{v}) + \Phi(\mathbf{u}) - \mathbf{u}^{\mathrm{T}}\mathbf{f} \tag{8}$$

One-step time integration

Letting $\alpha = \frac{t-t_0}{t_1-t_0}$ a one-step time integration scheme can be obtained introducing the approximation in time for u and v over the time step $[t_0, t_1]$

$$\begin{cases} \mathbf{v}(\alpha) = \mathbf{v}_0 + \alpha(\mathbf{v}_1 - \mathbf{v}_0) \\ \mathbf{u}(\alpha) = \mathbf{u}_0 + \alpha(\mathbf{u}_1 - \mathbf{u}_0) \end{cases} \tag{9}$$

The semi-discrete equations of motion (7) can be then rewritten in discrete form as

$$\begin{cases} \overline{\mathbf{v}} = \frac{\mathbf{u}_1 - \mathbf{u}_0}{\Delta \mathrm{t}} \\ \mathbf{M}\frac{\mathbf{v}_1 - \mathbf{v}_0}{\Delta \mathrm{t}} + \overline{\mathbf{s}}(\mathbf{u}) = \overline{\mathbf{p}} \end{cases} \tag{10}$$

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where $\overline{\mathbf{v}}, \overline{\mathbf{s}}$ and $\overline{\mathbf{p}}$ are representative value of $\mathbf{v}(\alpha)$, $\mathbf{s}(\alpha)$ and $\mathbf{p}(\alpha)$ over the step. For $\overline{\mathbf{v}}$ and $\overline{\mathbf{p}}$, a simple and natural choice is to define them as the integral mean of the corresponding function, so that we have

$$\begin{cases} \frac{\mathbf{v}_1+\mathbf{v}_0}{2} = \frac{\mathbf{u}_1-\mathbf{u}_0}{\Delta t} \\ \mathbf{M}\frac{\mathbf{v}_1-\mathbf{v}_0}{\Delta t} + \overline{\mathbf{s}}(\mathbf{u}) = \overline{\mathbf{p}} \quad \text{with} \quad \mathbf{p} \equiv \int_0^1 \boldsymbol{p}(\alpha)d\alpha \end{cases} \tag{10}$$

We can get $\mathbf{v}_1$ from the first of Eq. (11) and substitute it into the second, so obtaining the final form of the discrete equations of motion in terms of the only unknowns $\mathbf{u}_1$:

$$2\mathbf{M}\frac{\mathbf{u}_1+\mathbf{u}_0-\mathbf{v}_0\Delta t}{\Delta t^2} + \overline{\mathbf{s}}(\mathbf{u}) = -\mathbf{u}^{\mathrm{T}}\mathbf{f} \tag{8}$$

Different choices are possible for $\overline{\mathbf{s}}(\mathbf{u})$.

The new integration scheme

Now, we present a new one-step integration method aimed at preserving energy as well as the strain-displacement compatibility at the step end-point for generally nonlinear strain measures. Starting for the time interpolation in Eq.(9), the idea is to generalize the Simo and Tarnow method as follows. Firstly, at each spatial integration point, a representative value of the strain-displacement tangent operator is computed as integral mean of B(u) over the step:

$$\overline{\mathbf{B}} = \int_0^1 \mathbf{B}(\alpha)d\alpha \approx \sum_{n=1}^{N_t} \mathbf{B}(\mathbf{u}(\alpha_n))\,\bar{\omega}_n \tag{13}$$

Then the representative internal force vector of the step is evaluated as

$$\bar{\mathbf{s}}(\mathbf{u}) = \int_0^1 \overline{\mathbf{B}}^{\mathrm{T}}\boldsymbol{\sigma}dV \quad \text{with} \quad \boldsymbol{\sigma} = \mathbf{C}\frac{\boldsymbol{\varepsilon}(\boldsymbol{u}_1)+\boldsymbol{\varepsilon}(\boldsymbol{u}_0)}{\mathbf{2}} \tag{14}$$

with α_n and $\bar{\omega}_n$ temporal position and weight of the nth time integration point respectively. So, the scheme is very similar to Simo and Tarnow's one [2]. The difference is the integral mean $\overline{\mathbf{B}}$ instead of the mean value $\mathbf{B}_{\frac{1}{2}}$. This choice is the crucial idea to achieve energy conservation for arbitrarily nonlinear strain-displacement laws. Indeed, it is easy to verify that

$$\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_0 = \Delta t \int_0^1 \dot{\boldsymbol{\varepsilon}}d\alpha = \Delta t \int_0^1 \mathbf{B}\big(\mathbf{u}(\alpha)\big)\dot{\mathbf{u}}d\alpha = \overline{\mathbf{B}}\,(\mathbf{u}_1 - \mathbf{u}_0) \tag{15}$$

and then energy conservation holds independently of the strain measure nonlinearity:

$$\begin{aligned} &(\mathbf{u}_1 - \mathbf{u}_0)^{\mathrm{T}}\bar{\mathbf{s}}(\mathbf{u}) = \int_V (\mathbf{u}_1 - \mathbf{u}_0)^{\mathrm{T}}\,\overline{\mathbf{B}}^{\mathrm{T}}\,\mathbf{C}\frac{(\boldsymbol{\varepsilon}_1+\boldsymbol{\varepsilon}_0)}{\mathbf{2}}\mathrm{dV} \\ &= \frac{\mathbf{1}}{\mathbf{2}}\int_V \{\boldsymbol{\varepsilon}_1^{\mathrm{T}}\,\mathbf{C}\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_0^{\mathrm{T}}\,\mathbf{C}\boldsymbol{\varepsilon}_0\}\mathrm{dV} = \Phi(\mathbf{u}_1) - \Phi(\mathbf{u}_1) \end{aligned} \tag{15}$$

Clearly, the energy conservation tends to be exact by increasing the number of time integration points in (13). For example the Simo and Tarnow method, that is only approximately conserving for nonlinear measures other than quadratic, can be seen as a reduced integration of the proposed method where the integral mean of $\mathbf{B}$ is approximated by a single Gauss point in time. Although the number of time integration point for a converged $\overline{\mathbf{B}}$ is a-priori unknown for general problems, a few time points, as shown in the numerical examples, gives energy conservation for practical applications and time steps. Compared to the full integration of $\mathbf{s}(\mathbf{u})$ discussed in [1], the number of time integration points required is usually lower, because only $\mathbf{B}(\mathbf{u})$ in now integrated instead of the more nonlinear term $\mathbf{B}(\mathbf{u})^{\mathrm{T}}\boldsymbol{\sigma}(\mathbf{u})$. More importantly, in our proposal the internal forces are evaluated using only the stress at the end-points of the step, avoiding the inaccuracies caused by the inner stress derived from the displacement. With respect to the Sansour et al. method [3], the proposal fulfill exactly the strain-displacement relation at the step end-points and results to be unconditionally stable.

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Extension to nonlinear multi-body coupling laws

Multi-body coupling laws can be generically written as

$$\boldsymbol{\varepsilon}_c(\boldsymbol{u}) = \mathbf{0} \quad \forall \boldsymbol{x} \in \boldsymbol{\ell} \tag{17}$$

where ℓ denotes the boundary of the bodies where the coupling occurs. Without introducing additional DOFs for the Lagrange multipliers and upsetting the structure of existing finite element packages, general couplings expressed by Eq. (17) can be easily imposed in penalty form by adding to the strain energy of the system the penalty term

$$\Phi_c(\mathbf{u}_c) \equiv \int_\ell \frac{1}{2} \boldsymbol{\varepsilon}_c(\boldsymbol{u})^T \mathbf{C}_c \boldsymbol{\varepsilon}_c(\boldsymbol{u}) \, \mathrm{ds} \tag{18}$$

where $\boldsymbol{\varepsilon}_c(\boldsymbol{u})$ is interpreted as a pseudo-strain that is constrained to negligible values by a stiff pseudo-constitutive matrix $\mathbf{C}_c = a\,\hat{\mathbf{C}}_c$ with matrix $\hat{\mathbf{C}}_c$ chosen on the basis of the actual materi- als and geometry of the coupled bodies [6], so that the penalty factor a defines how small we want the penalty energy to be with respect to strain energy of the system. For general structural problems and coupling laws, the constraint tends to be satisfied point-wise with the mesh refinement as in the Lagrange multipliers approach for a sufficiently high penalty factor. The penalty coupling is useful to model real complex structures with generic interfaces and non- matching discretizations [5]. The gradient of $\Phi_c(\mathbf{u}_c)$ provides the equivalent internal forces due to the coupling

$$\boldsymbol{s}_c(\mathbf{u}) \equiv \int_\ell \mathbf{B}_c{}^T(\mathbf{u})\boldsymbol{\sigma} \, \mathrm{ds} \quad \text{with} \quad \boldsymbol{\sigma}_c = \mathbf{C}\boldsymbol{\varepsilon}_c(\mathbf{u}) \tag{19}$$

that has the same form of the element internal force vector with $\mathbf{B}_c(\mathbf{u})$ the tangent pseudo-strain/displacement operator. This is then assembled together with the element ones to obtain the global internal force vector to be used in global equations of motion Eq. (7). In this framework, the time integration method developed in the previous subsection can be directly applied to problems with generally nonlinear multi-body couplings, guarantying energy conservation and unconditional stability.

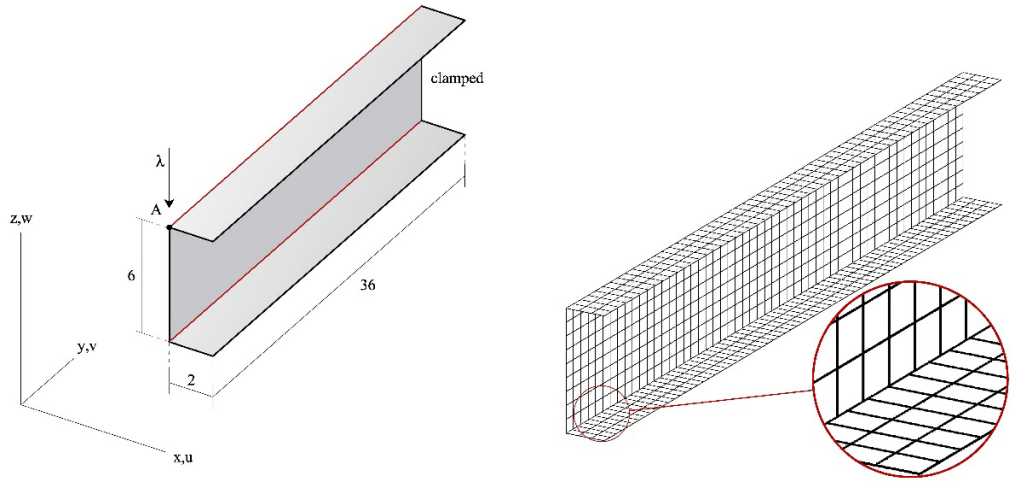

Figure 1: Thin-walled cantilever beam: geometry, boundary conditions and mesh

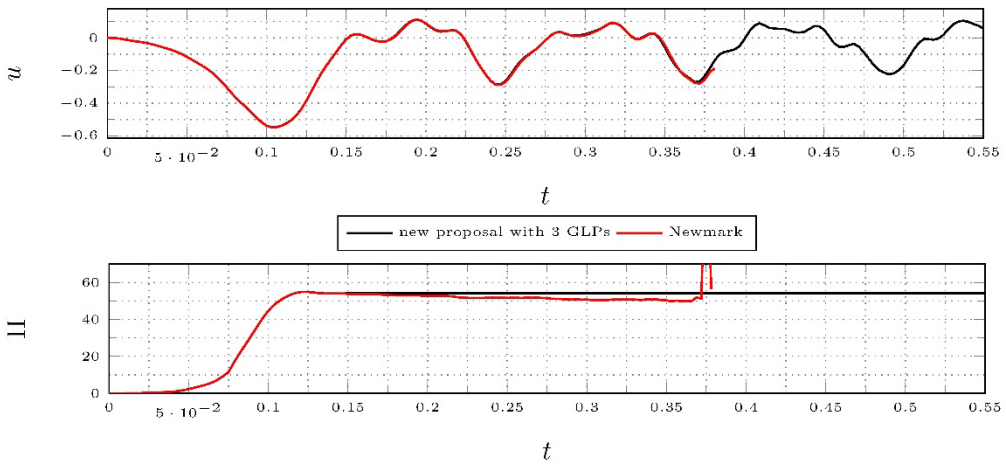

Figure 2: Thin-walled cantilever beam in composite material: Newmark vs new proposal

Numerical examples: thin-walled cantilever beam with local buckling

A thin-walled cantilever beam with a U cross section is considered in this test. Geometry, load, boundary conditions and mesh are reported in Fig. 1. Both isotropic and composite materials are considered. The isotropic material is characterized by $E = 1.0 \times 10^7$ and $\nu = 0.3$, while for the composite material we have $E_1 = 3.06 \times 10^7, E_2 = E_3 = 8.70 \times 10^6, \nu_{12} = 0.29$, $\nu_{23} = \nu_{13} = 0.3$ and $G_{12} = G_{13} = 3.24 \times 10^6, G_{23} = 2.90 \times 10^6$, with the material direction 1 corresponding to the longitudinal beam axis. The thickness of the walls is 0.05, while the density per unit of volume is 10^{-2}. The approximation is based on cubic NURBS functions. In such a structure, a patch coupling strategy is necessary for the Kirchhoff-Love shell model along the red interfaces. The load amplitude varies linearly from 0 to 150, a little higher value than the static buckling load, in 0.075 seconds. Afterwards, the load is removed linearly in 0.075 seconds and the simulation proceeds without load. The time step of the analysis is $\Delta t = 0.003125$ seconds. The results in terms of deflection of the loaded point and of total energy of the system are reported in Fig. 2 for the composite case. The Newmark method fails to preserve the energy after the load removal and its iterative solution fails when the energy oscillation becomes relevant. This is even more marked in the composite case with a dramatic blow-up. Instead, the proposal is perfectly stable with constant energy for a zero load and without any difficulty in the iterative solution.

Conclusions

A novel and very simple one-step time integration scheme for large deformation dynamics of elastic structures was presented. It is a generalization of the Simo and Tarnow method [2], designed to achieve unconditional stability for a quadratic strain, to arbitrarily nonlinear strain measures. The method is also suitable for models with finite 3D rotations, parametrized by the pseudo-rotation vector [9, 10].

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