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A mixed finite-element formulation for the elasto-plastic analysis of shell structures

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Abstract. Mixed assumed stress finite elements for elastic-perfectly plastic materials require the solution of a Closest Point Projection (CPP) involving all the element stress parameters for the integration of the constitutive equation. Here, a dual decomposition strategy is adopted to split the CPP at the element level into a series of CPPs at the integration points level and in a nonlinear system of equations over the element. The strategy is tested with a four nodes mixed shell finite element, named MISS-4, characterised by an equilibrated stress interpolation and a displacement field assumed only along its boundaries. The recovered elasto-plastic solution preserves all the advantages of MISS-4, namely it is accurate for coarse meshes in recovering the equilibrium path and evaluating the limit load showing a quadratic rate of convergence.

Introduction

Shell structures are largely used in different areas of engineering and constitute the primary elements of many mechanical, aerospace, marine, and civil constructions. Their nonlinear material behaviour plays a main role, for instance, in masonry and reinforced concrete structures [1] in presence of seismic actions. In such cases, and so in many others, an effective evaluation of the safety factor against plastic collapse [2] is of paramount importance to ensure reliable design choices and wise retrofitting strategies. The most frequently employed numerical approach for modelling shell structures is based on the Finite Element (FE) analysis. Beside more traditional displacement-based FEs, mixed formulations are nowadays widely employed thanks to their effectiveness in eliminating locking effects and improving accuracy. In mixed formulations, in fact, stresses or strains are assumed as independent variables together with displacements, and this gives the chance to design optimised FE characterised by high performance in both linear and nonlinear problems [3]. This, generally speaking, means low error in recovering the solution for coarse mesh grids, elevated rate of convergence and equivalent convergence properties for all the unknown fields. Recently, a wide number of mixed FE has been proposed [4]. Among these, a bright choice is represented by the so-called hybrid FEs where the assumed stress interpolation a-priori satisfies the equilibrium equations.

The family of FEs denoted with the acronym MISS (Mixed Isostatic Self Equilibrated Stresses) represents an interesting option among the state-of-art mixed hybrid FE [5, 6]. In particular, the element called MISS-4 is a quadrilateral four noded flat shell FE with 24 degrees of freedom (DOFs) [5]. It is developed within the Hellinger-Reissner variational formulation and is based on equilibrated and isostatic assumed stress fields. MISS-4 has demonstrated good performance in elasticity, being characterised by accurate solutions for coarse meshes, low sensitivity to mesh distortion, quadratic rate of converge for both displacement and stress fields.

In addition, an enhanced variant, called MISS-4c, improves the performance of MISS-4 [7] by using assumed stress fields that a-priori satisfy also compatibility equations for symmetric composite materials.

Despite being widespread in linear and geometrically nonlinear problems, mixed FEs based on assumed stresses are not a prior choice in incremental elastoplastic analysis, even though their aforementioned advantages hold. The reason for this resides in the fact that the weak imposition of the constitutive laws should be performed at the element level and the consequent return mapping process results into a Closest Point Projection (CPP) of the elastic predictor over the multi-surfaces admissibility domain of the element. This convex optimisation problem requires appropriate mathematical programming methods to be solved.

In this work we propose an alternative strategy for solving the CPP problem based on a dual decomposition method which splits the CPP over the element into two sub-problems. The first one is a series of standard CPP problems over the integration points (IP) which provide the IP stresses, while the second is a nonlinear system of equations over the element that recovers the assumed stress interpolation. The advantage of this procedure resides in the fact that the CPP problem at the IP level can be solved using the same strain-driven integration return map- ping schemes adopted by displacement-based formulations. The additional system of nonlinear equations can be readily solved by the Newton method. Within this strategy, it is possible to efficiently extend the field of application of MISS-4 FE to elasto-plastic problems, obtaining good accuracy even at coarse meshes thanks to the equilibrated assumed stress fields. In is worth noting that the proposed decomposed formulation furnishes the same discrete equations as in other works [8, 9, 10]. Further details on the proposed approach can be found in [11].

Elasto-plastic FE formulation

The FE used in this work is named MISS-4. We refer elsewhere [5, 7] for all the details on its formulation in the linear-elastic range. For the linear elastic problem, using a Hellinger-Reissner approach and the assumed displacement and generalised stresses interpolation one obtains

$$\int_{Be} \boldsymbol{t}^T \boldsymbol{\rho} d\Omega_e - \frac{1}{2}\int_{Be} \boldsymbol{t}^T \boldsymbol{F}\boldsymbol{t} d\Omega_e - L_{ext} = \boldsymbol{\beta}_e^T \boldsymbol{Q}_e \boldsymbol{d}_e - \frac{1}{2}\boldsymbol{\beta}_e^T \boldsymbol{H}_e \boldsymbol{\beta}_e - \boldsymbol{d}_e^T \boldsymbol{p}_e \tag{1}$$

where L_{ext} is the work of the element external loads pe, $\boldsymbol{H}_e$ and $\boldsymbol{Q}_e$ are the element compliance and the compatibility/equilibrium matrices, $\boldsymbol{t}$ and $\boldsymbol{\rho}$ are the generalised stresses and strains, re-spectively, $\boldsymbol{\beta}_e$ and $\boldsymbol{d}_e$ are the stress and displacement parameters, respectively, and $\boldsymbol{F}$ is the inverse of the first order shear deformation theory elasticity matrix. The key problem of an elasto-plastic analysis is represented by the integration of the constitutive equations. A common approach is based on the use of a Backward-Euler integration scheme of the incremental constitutive equations, where the stresses become an implicit function of the assigned strain (displacement) increment. For mixed FE, the weak form of the finite step constitutive relations over the element becomes

$$\begin{cases} \boldsymbol{H}_e\left(\boldsymbol{\beta}_e - \boldsymbol{\beta}_e^{(n)}\right) + \sum_g \mu_g \dfrac{\partial f_g}{\partial \boldsymbol{\beta}_e} = \boldsymbol{Q}_e \Delta \boldsymbol{d}_e \\ \mu_g \geq 0, \;\; f_g \leq 0, \;\; \mu_g f_g = 0 \end{cases} \tag{2}$$

where we assume to test the plastic admissibility conditions in a discrete number of IP identified by the subscript g, $\boldsymbol{\Delta} = (\cdot)^{(\boldsymbol{n+1})} - (\cdot)^{(\boldsymbol{n})}$ represents the difference between quantities at step $\boldsymbol{n} + \boldsymbol{1}$ and $\boldsymbol{n}$, $\boldsymbol{f}$ is the yield function, and $\boldsymbol{\mu_g}$ are the positive plastic multipliers. The superscript $(+\boldsymbol{1})$ is omitted to simplify the notation.

It is easy to show that Eq. (2) are the first order conditions of the following convex optimisation problem

$$\begin{aligned} &\min_{\boldsymbol{\beta}_e} && \frac{1}{2}(\boldsymbol{\beta}_e - \boldsymbol{\beta}_e^*)^T \boldsymbol{H}_e (\boldsymbol{\beta}_e - \boldsymbol{\beta}_e^*) \\ &\mathit{subject\ to} && f_g = f\left[\boldsymbol{N}_{tg}\boldsymbol{\beta}_e\right] \leq 0, \;\; \forall g \end{aligned} \tag{3}$$

which represents a CPP of the predictor $\boldsymbol{\beta}_e^* = \boldsymbol{\beta}_e^{(n)} + \boldsymbol{H}_e^{-1}\boldsymbol{Q}_e\Delta\boldsymbol{d}_e$ over the admissible domain of the element which, in this case, is a multi-surface yield domain. The solution of this problem is unique when the quantities on the initial step are known and $\Delta\boldsymbol{d}_e$ assigned.

Equations (2) furnish the implicit constitutive step equation we are searching for as $\beta_e = \boldsymbol{\beta}_e[\boldsymbol{\beta}_e^{(n)}, \Delta\boldsymbol{d}_e]$. The internal force vector of the element, for a given $\Delta\boldsymbol{d}_e$ and starting from a known state $\boldsymbol{\beta}_e^{(n)}$, is expressed as

$$\boldsymbol{s}_e[\boldsymbol{d}_e] = \boldsymbol{Q}^T\boldsymbol{\beta}_e\left[\boldsymbol{\beta}_e^{(n)}, \Delta\boldsymbol{d}_e\right] \tag{4}$$

while the tangent stiffness matrix becomes

$$\boldsymbol{K}_{et} = \frac{\partial \boldsymbol{s}_e[\boldsymbol{d}_e]}{\partial \boldsymbol{d}_e} = \boldsymbol{Q}_e^T\boldsymbol{H}_{et}^{-1}\boldsymbol{Q}_e, \qquad \boldsymbol{H}_{et}^{-1} = \frac{\partial \boldsymbol{\beta}_e}{\partial \boldsymbol{\eta}_e} \tag{5}$$

And $\boldsymbol{\eta}_e = \boldsymbol{Q}_e\boldsymbol{d}_e$ is the generalized strain.

The incremental elastoplastic analysis

The equilibrium path of the structures is usually evaluated by means of a continuation method [11]. Starting from a known equilibrium point $z_0 = z^{(n)}$, where $z = [d, \lambda]$, a new state z^{k+1} is obtained by solving, through a Newton iteration, the following system representing the equilibrium equations plus the arc-length constraint for an assigned value of $\Delta\xi$.

$$\begin{cases} \boldsymbol{r}[\boldsymbol{d},\lambda] = \boldsymbol{s}[\boldsymbol{d}] - \lambda\boldsymbol{p} = \boldsymbol{0} \\ r_\lambda[\boldsymbol{d},\lambda] = \boldsymbol{n}_z^T\boldsymbol{M}_z(\boldsymbol{z} - \boldsymbol{z}_0) - \Delta\xi = 0 \end{cases} \quad with \;\; \Delta\xi = \boldsymbol{n}_z^T\boldsymbol{M}_z(\boldsymbol{z}_1 - \boldsymbol{z}_0), \tag{6}$$

where $\boldsymbol{z}_1$ is the first predictor. The most used and effective arc-length equation is a moving hyperplane with $\boldsymbol{M}_z = diag(\boldsymbol{M}, \mu)$ a suitable metric matrix and $\boldsymbol{n}_z = [\boldsymbol{n}, n\_\lambda]$ its normal vector. Letting

$$\boldsymbol{K} = \frac{\partial \boldsymbol{s}[\boldsymbol{d}]}{\partial \boldsymbol{d}} \tag{7}$$

evaluated as assemblage of element matrices $\boldsymbol{K}_{et}$ in Eq. (5), Newton iterations are applied to the extended nonlinear system of equations (6), giving a sequence of estimates

$$\Delta\boldsymbol{z}_{j+1} = \Delta\boldsymbol{z}_j + \dot{\boldsymbol{z}}_j \;\; and \;\; \boldsymbol{z}_{j+1} = \boldsymbol{z}_0 + \Delta\boldsymbol{z}_j$$

Where the correction $\dot{\boldsymbol{z}}_j$ is evaluated as

$$\begin{cases} \boldsymbol{K}_j\dot{\boldsymbol{d}} - \dot{\lambda}\,\boldsymbol{p} \; - \boldsymbol{r}_j \\ \boldsymbol{n}^T\boldsymbol{M}\dot{\boldsymbol{d}} + \mu n_\lambda\dot{\lambda} = -r_{\lambda j} \end{cases}$$

At each iteration, the internal forces vector s_j, and then the residual r_j, is evaluated by assembling element quantities $s_e^j = Q^T\beta_e[\beta^{(n)}, \Delta d_e^j]$ with the stresses obtained using the CPP problem in Eq. (3).

The solution of the Closest Point Projection scheme over the element

The mathematical programming problem in Eq.(3) represents the main difference between mixed and displacement-based FE formulations.

Herein, the CPP problem is alternatively solved by means of a dual decomposition technique. The dual decomposition produces a series of CPP over the IPs, as in usual displacement formulations, therefore simplifying the solution which can be performed using standard numerical tools based on the application of the Newton method.

The dual decomposition of the CPP problem over the element

Two kind of dual decomposition approaches are investigated, depending on the way the plastic admissibility is checked. In a first case, we assume to express the plastic-admissibility condition in terms of the generalised stresses $\boldsymbol{t}$

$$f_g[\boldsymbol{t}_g] = f[\boldsymbol{t}_g, \boldsymbol{x}_g] \leq 0, \ \forall g \tag{8}$$

and to control it in a finite number of IPs having middle-surface coordinate x_g and generalised stresses $\boldsymbol{t}_g = \boldsymbol{N}_t[\boldsymbol{x}_g]\boldsymbol{\beta}_e$. The decomposition process is performed at the level of these midplane shell IPs.

The second approach further decomposes the problem at level of material point. In this case, the plastic admissibility is expresses by standard yield functions in terms of the Cauchy 3D stresses $\boldsymbol{\sigma}$ evaluated in a discrete number of control points distributed also in the thickness direction $(\boldsymbol{x}_g, z_{gm})$

$$f_{gm}[\boldsymbol{\sigma}_{gm}] = f[\boldsymbol{\sigma}_{gm}, \boldsymbol{x}_g, z_m] \leq 0 \ \ \forall g, m \tag{9}$$

Where $\boldsymbol{\sigma}_{gm} = \boldsymbol{E}[\boldsymbol{z}_m]^T \boldsymbol{N}_t[\boldsymbol{x}_g]\boldsymbol{\beta}_e$ is the stress at the $(g, m)IP$.

A dual decomposition scheme for the CPP problem over the element

Assuming a numerical integration scheme to evaluate the complementary energy terms so that

$$t_g = N_{tg}\beta_e \rightarrow \sum_g t_g^T F t_g w_g = \beta_e^T H_e \beta_e \tag{10}$$

The minimization problem in Eq.(3) becomes

$$\min_{\boldsymbol{t}_e} \quad \frac{1}{2}\sum_g \left(\boldsymbol{t}_g - \boldsymbol{t}_g^{(n)}\right)^T \boldsymbol{F}\left(\boldsymbol{t}_g - \boldsymbol{t}_g^{(n)}\right) w_g - \boldsymbol{\beta}_e^T \boldsymbol{Q}_e \Delta \boldsymbol{d}_e$$
$$subject\ to \ \ f_g[t_g] \leq 0, \ \forall g = 1, \dots, N_g, \qquad t_g = N_{tg}\beta_e \tag{11}$$

where the objective function of problem in Eq.(3), since a constant with βe term is inessential, is rewritten as

$$\frac{1}{2}\left(\boldsymbol{\beta}_e - \boldsymbol{\beta}_e^{(n)}\right)^T \boldsymbol{H}_e\left(\boldsymbol{\beta}_e - \boldsymbol{\beta}_e^{(n)}\right) - \boldsymbol{\beta}_e^T \boldsymbol{Q}_e \Delta \boldsymbol{d}_e \tag{12}$$

The Lagrangian associated to problem in Eq.(11), under the assumption $\mu_g \geq 0$, is

$$L = \frac{1}{2}\sum_g \left(\boldsymbol{t}_g - \boldsymbol{t}_g^{(n)}\right)^T \boldsymbol{F}\left(\boldsymbol{t}_g - \boldsymbol{t}_g^{(n)}\right) w_g - \boldsymbol{\beta}_e^T \boldsymbol{Q}_e \Delta \boldsymbol{d}_e + \sum_g \mu_g f_g[\boldsymbol{t}_g] w_g - \sum_g \Delta\boldsymbol{\rho}_g^T (\boldsymbol{t}_g - \boldsymbol{N}_{tg}\boldsymbol{\beta}_e) w_g$$

Having relaxed the element interpolation equations, the first order equations can be subdivided into those involving only IPs variables and those regarding elemental quantities. The firsts are

$$\frac{\partial L}{\partial \boldsymbol{t}_g} \rightarrow \begin{cases} \boldsymbol{F}\left(\boldsymbol{t}_g - \boldsymbol{t}_g^{(n)}\right) + \mu_g \frac{\partial f_g[\boldsymbol{t}_g]}{\partial \boldsymbol{t}_g} - \Delta\boldsymbol{\rho}_g = \boldsymbol{0} \\ \mu_g f_g[\boldsymbol{t}_g] = 0 \ \ \mu_g \geq 0 \ \ f_g[\boldsymbol{t}_g] \leq 0 \end{cases} \ \forall\, g, \tag{13}$$

while the equations regarding elemental variables are

$$\boldsymbol{r}_g = \frac{\partial L}{\partial \boldsymbol{\rho}_g} = \boldsymbol{t}_g - \boldsymbol{N}_{tg}\boldsymbol{\beta}_e = \boldsymbol{0} \ \ \forall g$$
$$\boldsymbol{r}_e = \frac{\partial L}{\partial \boldsymbol{\beta}_e} = \boldsymbol{Q}_e \Delta \boldsymbol{d}_e - \sum_g \boldsymbol{N}_{tg}^T \Delta\boldsymbol{\rho}_g w_g = \boldsymbol{0} \tag{14}$$

A sequence of IP state determination and element state determination problems allows the evaluation of a plastically-admissible $\boldsymbol{\beta}_e$ for an assigned $\boldsymbol{\Delta d}_e$, as explained in [11].

Numerical results

The proposed FE solution for the elasto-plastic analysis of shell structures is herein tested in a numerical example. The results obtained with MISS-4 are compared with those provided by the

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commercial software Abaqus using S4 and S8r FEs. The plastic admissibility is tested using a grid of 2 × 2 Gauss Points GP grid over the FE mid-plane and m = 8 GP along the thickness. In all tests the material is elastic perfectly plastic and a von Mises yield criterion is assumed. The test regards a the square plate of length L. The thickness is h = L/100. Due to the symmetry of the problem, only a quarter of the plate is analysed. Two boundary conditions are studied, namely simply supported and clamped where in both cases the normal rotation to the side is constrained. A uniformly distributed out-of-plane load is applied, having amplitude $p = \lambda M_0/(2L)^2$, with $M_0 = \sigma_0 t^2/4$ and σ_0 is Mises stress limit.

Table 1: Square plate: limit load for different models and mesh densities, clamped condition.

FE	*m*	mesh density				
		2×2	4×4	8×8	16×1	32×3
MISS-4	8	44.164	46.146	46.624	46.215	45.643
MISS-4	g	43.664	45.629	46.104	45.705	45.143
Abaqus S4	8	61.975	54.797	49.840	47.267	45.965

The limit loads for different FE and meshes are shown in Tables 1 for the clamped case. It shows that MISS-4 is accurate also for the coarsest mesh. The good convergence properties of MISS-4 are highlighted in Fig. 1 which shows that for both the boundary conditions it gives lower error than Abaqus S4.

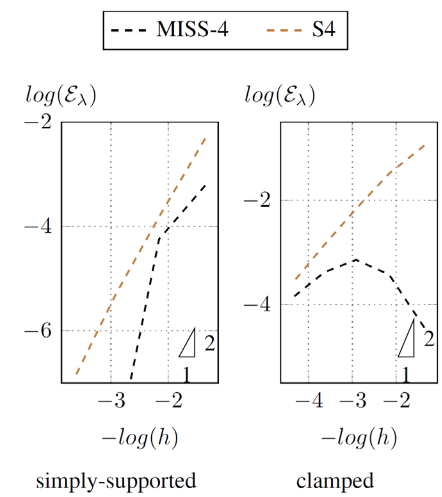

Figure 1: Square plate: convergence of the limit load.

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