Materials Research Proceedings **26** (2023) 163-168 https://doi.org/10.21741/9781644902431-27

Stress and strain fields in non-prismatic inhomogeneous beams

Giovanni Migliaccio

Department of Engineering and Architecture, University of Parma, Parma, Italy

giovanni.migliaccio.it@gmail.com

Keywords: Analytical Solutions, Numerically Efficient Method, Large Displacements

Abstract. Beamlike structures are commonly studied via 1D beam models, which are more efficient than 3D finite element methods, but do not permit accurate predictions of 3D stresses in non-prismatic cases. Despite the progress made in their modeling, either via direct 1D approaches or dimensional reductions from 3D formulations, the accurate analytical prediction of stresses and strains in beamlike yet 3D elements, with non-uniform properties both in terms of materials and cross-section shape, subject to large displacements, is an open problem. This work presents a model for such elements that is particularly suitable for efficient numerical implementations and that allows accurate analytical predictions of stresses and strains. A paradigmatic example shows the importance of non-trivial stress terms that are absent in prismatic homogeneous elements and the inadequacy of usual beam models and stepped-beam approaches when dealing with predictions of stresses and strains in non-prismatic inhomogeneous cases.

Introduction

The analytical prediction of stresses and strains in non-prismatic beamlike elements is very challenging. In the study of helicopter rotor blades, for instance, one needs to properly account for non-trivial couplings among bending, torsion, and traction associated with the cross-sectional pre-twist [1-3]. In addition to this, further complexities are present in wind turbine blades, which are also tapered [4-5]. Not to mention the complexities associated with large displacements and complex material properties. However, analytical predictions of stresses and strains are not simple even in prismatic homogenous isotropic cases. A paradigmatic example is represented by the classical Saint-Venant's problem [6-7], which is often addressed via approximated methods because of the difficulty to find closed-form solutions and the need for application-oriented formulas for cases of engineering interest. Jourawski's formula is one such application-oriented solutions [8], but it holds only for linear elastic prismatic beams whose material is homogeneous and isotropic, and provides erroneous predictions in non-prismatic cases [3,9,10].

Following Jourawski's method, several scholars have proposed formulas for shear stresses in tapered beams, e.g., Bleich [11], Pugsley and Weatherhead [12], Krahula [13], Bertolini et al. [14], Balduzzi et al. [10]. A review about critical issues and deficiencies of current engineering methods when dealing with tapered beams can be found in the works of Paglietti and Carta [9], Balduzzi et al. [10], and Migliaccio et al. [15]. Literature works usually address tapered beams undergoing small displacements, assume Navier's formula for the cross-sectional normal stresses, and derive formulas for the corresponding shear stresses by the static equilibrium of a beam slice in its reference (undeformed) state. Migliaccio's works [5,15,16] do not rely on such assumptions, account for the effects of large displacements, and is the basis of the present study.

Here, after introducing a mechanical model tailored to non-prismatic inhomogeneous beams susceptible to large deflections, an analytical method is proposed to accurately predict their stress and strain fields. Finally, the inadequacy of usual beam models when dealing with predictions of stresses and strains in the considered elements is discussed.

Mechanical model

The present beamlike body is a set of deformable plane figures (cross-sections) attached at a deformable line (center-line). For convenience, cross-sections in the reference state are orthogonal to the center-line. Changes from the reference state to the actual (deformed) state may take place with center-line large displacements and small strains, while cross-sections may undergo warping displacements, in and out of plane, which produce small deformation. See Figure 1.

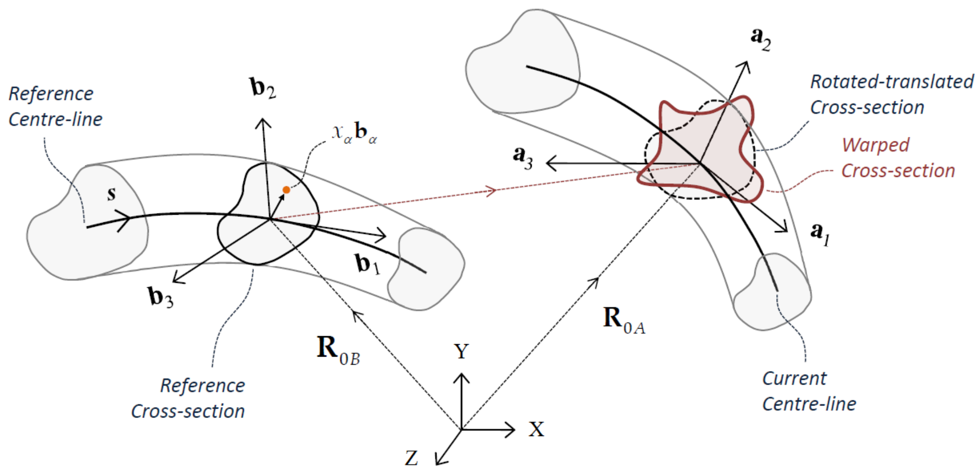

Figure 1: Schematic reference (left) and actual (right) states of non-prismatic beamlike element.

Using summation convention, with Latin (Greek) indices taking values on 1, 2, 3 (2, 3), the positions of the body points in the reference state, $\mathbf{R}_B$, and in the actual state, $\mathbf{R}_A$, are written as

$$\begin{aligned} \mathbf{R}_B(z_i) &= \mathbf{R}_{0B}(z_1) + x_\alpha(z_i)\mathbf{b}_\alpha(z_1), \\ \mathbf{R}_A(z_i,t) &= \mathbf{R}_{0A}(z_1,t) + x_\alpha(z_i)\mathbf{a}_\alpha(z_1,t) + w_k(z_i,t)\mathbf{a}_k(z_1,t), \end{aligned} \tag{1}$$

where $\mathbf{R}_{0B}$ and $\mathbf{R}_{0A}$ identify the center-line points in the reference and actual states with respect to a fixed Cartesian frame X,Y,Z; $\mathbf{b}_i=\mathbf{b}_i(s)$ is the local triad of orthogonal unit vectors in the reference state; $\mathbf{b}_1$ is tangent to the reference center-line of arc-length s; $\mathbf{a}_i=\mathbf{a}_i(s,t)$ is the image $\mathbf{b}_i$ in the current state and generally depends on the arc-length s and time t; x_α identify the cross-section points; w_i are warping displacements; finally, z_i are time-independent variables, with $z_1=s$ and z_α spanning a two-dimensional domain mapping the positions of the cross-section points. In this work, $x_i=\Lambda_{ij}z_j$, $\Lambda_{11}=1$, $\Lambda_{22}=\Lambda_2(z_1)$, $\Lambda_{33}=\Lambda_3(z_1)$, and coefficients Λ_{ij} with i≠j identically vanish.

The body strain state is described via the Green-Lagrange strain tensor $\mathbf{E}$ and the vector fields $\boldsymbol{\gamma} = \mathbf{T}^T\mathbf{R}'_{0A} - \mathbf{R}'_{0B}$ and $\mathbf{k} = \mathbf{T}^T\mathbf{k}_A - \mathbf{k}_B$, where $\mathbf{k}_B$ and $\mathbf{k}_A$ are curvature vectors in the reference and actual states, $\mathbf{T} = \mathbf{a}_i \otimes \mathbf{b}_i$, $\otimes$ is tensor product, and prime denotes s-derivative. Specifically, for small strain and warping fields, $\mathbf{E}$ is written as in [16], i.e., $2\mathbf{E} = \mathbf{T}^T\mathbf{H}+\mathbf{H}^T\mathbf{T}-2\mathbf{I}$, where $\mathbf{H}$ is the derivative of $\mathbf{R}_A$ with respect to $\mathbf{R}_B$ (deformation gradient) and $\mathbf{I}$ is the identity tensor.

The body stress state is described via the second (symmetric) Piola-Kirchhoff stress tensor $\mathbf{S} = S_{ij}\,\mathbf{b}_i \otimes \mathbf{b}_j$, which is given in terms of $\mathbf{E} = E_{ij}\,\mathbf{b}_i \otimes \mathbf{b}_j$ via a linear elastic constitutive model,

$$\mathbf{S} = \frac{Y}{1+\nu}\mathbf{E} + \frac{\nu Y}{(1+\nu)(1-2\nu)} tr\mathbf{E}\,\mathbf{I}, \tag{2}$$

where Y and ν are Young's modulus and Poisson's ratio, respectively. For completeness, it is useful introducing also the Cauchy stress tensor, $\mathbf{C} = C_{ij}\,\mathbf{a}_i \otimes \mathbf{a}_j$, which is related to $\mathbf{S}$ via the deformation gradient $\mathbf{H}$ and is associated with the stress in the body actual state [3,16].

Body stresses and strains can be obtained as solutions of balance equations (partial differential equations with boundary conditions) derivable from the principle of virtual power. To this aim, we introduce the external power functional Π_e (to describe the interactions between the body and the external environment), and the internal power functional Π_i (to describe the interactions among the body parts), as follows

$$\Pi_e = \int_V \mathbf{b}\cdot\mathbf{v} + \int_{\partial V} \mathbf{p}\cdot\mathbf{v}, \quad \Pi_i = \frac{d}{dt}\int_V \Phi, \tag{3}$$

where **b** represents body loads per unit body reference volume V, **p** denotes surface actions per unit area of the boundary ∂V, **v** is the time rate of the actual position of the body points, and the strain energy density $\Phi=\mathbf{S}\cdot\mathbf{E}/2$ is half the scalar product of **S** and **E**. Balance equations are finally obtainable by enforcing that for any velocity field attainable by the body its interactions with the external environment and among its parts are such that the total power vanishes (i.e., $\Pi_e=\Pi_i$) at any value of the evolution parameter t.

Following [16], and studying inhomogeneous beams with tapered, pre-twisted cross-sections and external actions (as in a Saint-Venant's beam) applied only at the end cross-sections, we write the cross-sectional strains, E_{11}, $E_{\alpha 1}$, in the form

$$\begin{aligned}
&E_{11} = k_2 x_3 - k_3 x_2 + \gamma_1 + e_{1,1} + k_{B1}(x_3 e_{1,2} - x_2 e_{1,3}),\\
&2E_{21} = e_{1,2} - k_1 x_3 + e_2 + 2(1+\nu)(k_2 x_3 - k_3 x_2 + \gamma_1)(\Lambda_2^{-1}\Lambda_2' x_2 - k_{B1} x_3),\\
&2E_{31} = e_{1,3} + k_1 x_2 + e_3 + 2(1+\nu)(k_2 x_3 - k_3 x_2 + \gamma_1)(\Lambda_3^{-1}\Lambda_3' x_3 + k_{B1} x_2),
\end{aligned} \tag{4}$$

where E_{11} and $E_{\alpha 1}$ are normal and shear strains in the cross-sections, γ_1 is center-line extension, k_α and k_1 are bending and torsion curvatures, k_{B1} is pre-twist function, Λ_α are taper functions, comma denotes x_i-derivate, and the scalar fields e_1, e_2, e_3 are solutions of the PDEs problem

$$\begin{aligned}
&e_{1,22} + e_{1,33} = 0 \quad in\ \Sigma,\\
&e_{2,2} + e_{3,3} = d_{k_3'} k_3' + d_{k_3} k_3 + d_{k_2'} k_2' + d_{k_2} k_2 \quad in\ \Sigma,\\
&e_{3,2} - e_{2,3} = g_{k_3'} k_3' + g_{k_3} k_3 + g_{k_2'} k_2' + g_{k_2} k_2 + g_{\gamma_1}\gamma_1 \quad in\ \Sigma,\\
&(e_{1,2} - k_1 x_3) n_2 + (e_{1,3} + k_1 x_2) n_3 = 0 \quad on\ \partial\Sigma,\\
&e_2 n_2 + e_3 n_3 = 0 \quad on\ \partial\Sigma.
\end{aligned} \tag{5}$$

In (5), Σ and $\partial\Sigma$ are cross-sectional domain and its boundary, n_α are components of the outward unit normal on $\partial\Sigma$, Λ_Y is the ratio between the material Young's modulus Y at a generic s and its reference value at s=0, and coefficients $d_{(\cdot)}$ and $g_{(\cdot)}$ are defined as follows

$$\begin{aligned}
&d_{k_3'} = 2(1+\nu)x_2,\\
&d_{k_3} = 2(1+\nu)\left(\Lambda_3^{-1}\Lambda_3' + 2\Lambda_2^{-1}\Lambda_2' + \Lambda_Y^{-1}\Lambda_Y'\right)x_2,\\
&d_{k_2'} = -2(1+\nu)x_3,\\
&d_{k_2} = -2(1+\nu)\left(\Lambda_2^{-1}\Lambda_2' + 2\Lambda_3^{-1}\Lambda_3' + \Lambda_Y^{-1}\Lambda_Y'\right)x_3,\\
&g_{\gamma_1} = 2k_{B1}(2+2\nu),\\
&g_{k_3'} = -2\nu x_3,\\
&g_{k_3} = -2(1+\nu)\Lambda_3^{-1}\Lambda_3' x_3 - 2k_{B1}(3+2\nu)x_2,\\
&g_{k_2'} = -2\nu x_2,\\
&g_{k_2} = -2(1+\nu)\Lambda_2^{-1}\Lambda_2' x_2 + 2k_{B1}(3+2\nu)x_3.
\end{aligned} \tag{6}$$

Equations (4)-(6) show that the cross-sectional strains E_{i1} explicitly depend on the geometric and material characteristics of the body ($\Lambda_\alpha, k_{B1}, \Lambda_Y$) and can be expressed as linear combinations of the 1D strains γ_1 and k_i and their s-derivative. This result is interesting as it allows us to consider

Materials Research Proceedings **26** (2023) 163-168 https://doi.org/10.21741/9781644902431-27

the individual effect of each 1D strain by solving PDEs whose solution only depends on the shape of the cross-sectional domain Σ. Such PDEs admit closed-form solutions only in a few cases (e.g., [3,16]), but can always be solved numerically. However, apart from directly determining the scalar fields e_i and the relevant strains and stresses, we can obtain analytical information on these latter without explicitly solving the PDEs (5)-(6), as is discussed in the following section.

Analytical results

Equations (4)-(6), together with (2), can be used to derive a closed-form expression of the flow of shear stress through the cross-sectional chords of non-prismatic inhomogeneous beams. To show this, let us consider the generic cross-section in Figure 2 (left): the boundary of the dashed domain Σ_q is oriented counter-clockwise and is composed of internal lines $\partial\Sigma_i$ and external lines $\partial\Sigma_e$, whose outward normal **n** and tangent **t** have components n_α and t_α, respectively.

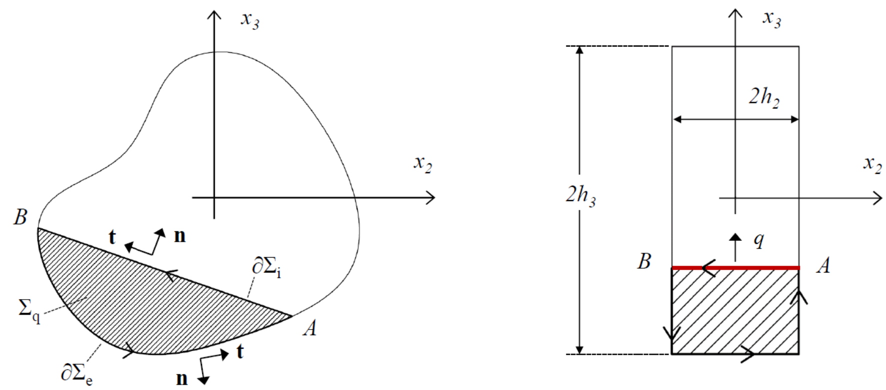

Figure 2: Generic reference cross-section of a beamlike solid (left) and rectangular shape (right).

The cross-sectional shear flow q through the internal lines $\partial\Sigma_i$ is defined via the line integral

$$q = \int_{\partial\Sigma_i} S_{1\alpha} n_\alpha \,, \tag{7}$$

and, by using (2), (4)-(6), and integration methods based on Green's formulas, takes the form

$$q = -YS_2 k_2' + YS_3 k_3' - YZ_2 k_2 + YZ_3 k_3 + YZ_1 \gamma_1 \,, \tag{8}$$

where coefficients S_α and Z_i are defined by the following surface and line integrals

$$S_2 = \int_{\Sigma_q} x_3, \quad S_3 = \int_{\Sigma_q} x_2 \,, \tag{9}$$

$$\begin{aligned} Z_2 &= \int_{\partial\Sigma_e} x_3 (\Lambda_2^{-1}\Lambda_2' x_2 n_2 + \Lambda_3^{-1}\Lambda_3' x_3 n_3) + k_{B1} \int_{\partial\Sigma_i} x_3 x_\alpha t_\alpha + \Lambda_Y^{-1}\Lambda_Y' \int_{\Sigma_q} x_3, \\ Z_3 &= \int_{\partial\Sigma_e} x_2 (\Lambda_2^{-1}\Lambda_2' x_2 n_2 + \Lambda_3^{-1}\Lambda_3' x_3 n_3) + k_{B1} \int_{\partial\Sigma_i} x_2 x_\alpha t_\alpha + \Lambda_Y^{-1}\Lambda_Y' \int_{\Sigma_q} x_2, \\ Z_1 &= \int_{\partial\Sigma_i} \Lambda_2^{-1}\Lambda_2' x_2 n_2 + \Lambda_3^{-1}\Lambda_3' x_3 n_3 + k_{B1} \int_{\partial\Sigma_e} x_\alpha t_\alpha . \end{aligned} \tag{10}$$

Equations (8)-(10) show the explicit dependence of the shear flow q on the body geometric and material characteristics (taper functions Λ_α, pre-twist function k_{B1}, material function Λ_Y). Note that coefficients S_α are classical area moments and are present also in prismatic cases. On the contrary, coefficients Z_i account for the effects of taper, pre-twist, and material inhomogeneity, which are absent in prismatic homogeneous cases and are unpredictable via usual linear theories of prismatic homogenous beams. Specifically, the shear formula (8) is a generalization of results derivable from the classical Saint-Venant's theory (such as Jourawski's formula), as well as of formulas presented in recent works [5,15]. Moreover, it is worth noting that if the s-derivative of the material and taper functions vanish, together with the pre-twist function (i.e., in the prismatic homogenous case), the cross-sectional shear flow q will depend only on the s-derivative of the bending curvatures k_α via the area moments S_α. Furthermore, if the actual and reference states of the body can be considered almost coincident for equilibrium purposes (small displacements), the s-derivative of the bending curvatures k_α turn out to be proportional to the transverse shear forces calculated in the reference state of the body and equation (8) reduces exactly to the classical Jourawski's solution [8].

The shear formula (8) can be used for analytical predictions of shear stresses in thin-walled non-prismatic beams with spanwise variable material properties in the same way as Jourawski's formula is used in prismatic homogenous cases. We show this via a paradigmatic example in which the cross-sectional shape is rectangular, as in the original Jourawski's work [8], but its dimensions are varied along the beam length, together with the material properties.

Variable dimensions and material properties: a paradigmatic example.
Let us consider a straight beam with rectangular cross-sections, which are tapered from the root to the tip according to the taper functions $\Lambda_2=\Lambda_3=\Lambda(s)$. Let us assume that the material properties can vary as well: specifically, the Young's modulus is given in the form $Y(s)=Y_0\Lambda_Y(s)$, where Y_0 is the Young's modulus at the root section (clearly, for constant material properties $\Lambda_Y(s)=1$).

To illustrate the analytical procedure for predicting stresses and strains in elements of this kind, let us consider the case in which this beam is fixed at the root (s=0) and is bent within the plane x_1-x_3 by a tip load applied in the same plane. In such case, coefficients S_2, Z_2, Z_1 are

$$S_2 = h_2(x_3^2 - h_3^2), \quad Z_2 = h_2(x_3^2 - 3h_3^2)\Lambda^{-1}\Lambda', \quad Z_1 = 2h_2x_3\Lambda^{-1}\Lambda', \tag{11}$$

and the cross-sectional shear flow, from (8) and (11), turns out to be expressible in the form

$$\frac{q}{2h_2Y} = -\frac{x_3^2 - h_3^2}{2}k_2' - \frac{\Lambda'}{\Lambda}\left(\frac{x_3^2 - 3h_3^2}{2}k_2 - x_3\gamma_1\right) - \frac{\Lambda_Y'}{\Lambda_Y}\frac{x_3^2 - h_3^2}{2}k_2, \tag{12}$$

where $2h_2(s)$, $2h_3(s)$ are the dimensions of the rectangular cross-section in Figure 2 (right) at the axial coordinate s, and $q/(2h_2)$ represents the mean shear stress orthogonal to the chord AB.

As is apparent, equation (12) extends the well-known Jourawski's solution [8]: the mean shear stress over a chord parallel to the cross-section width $2h_2$ consists of a first Jourawski-like term (proportional to the s-derivative of the bending curvature k_2), plus additional terms proportional to the s-derivative of taper and material functions, which are needed to accurately predict the shear stresses in non-prismatic inhomogeneous cases, as is also confirmed by numerical analyses for the present beam shape (not reported here for brevity), as well as from the analyses performed on other beam shapes in recent works (e.g., [3,5,15,16]).

Conclusions

Non-prismatic inhomogeneous beams undergoing large displacements are characterized by stress distributions that can be quite different from those predictable by linear theories of prismatic homogeneous beams. The approach proposed in this work allows accurate analytical predictions of stresses and strains in such elements.

In a paradigmatic example, we have shown the importance of geometric and material functions that produce non-trivial stress distributions, which are unpredictable via usual beam models and which confirm the inadequacy of stepped-beam approaches when dealing with predictions of stresses and strains in non-prismatic inhomogeneous cases. These results are also corroborated by several numerical analyses, some of which are reported in recent works by the author.

The results of this paper focus on the cross-sectional out-of-plane deformations and the relevant stress fields. Investigations about the in-plane deformations and the associated stress fields, as well as analytical studies about the influence of other important parameters (e.g., an initial curvature of the beam), will be addressed in subsequent extended works.

References

[1] Krenk S., The torsion-extension coupling in pretwisted elastic beams, Int. J. Solids Struct., 19, 67-72, 1983. https://doi.org/10.1016/0020-7683(83)90038-0

[2] Rosen A., Structural and dynamic behavior of pretwisted rods and beams, ASME, 44, 483-515, 1991. https://doi.org/10.1115/1.3119490

[3] Migliaccio G., Ruta G., The influence of an initial twisting on tapered beams undergoing large displacements, Meccanica, 56, 7, 1831-1845, 2021. https://doi.org/10.1007/s11012-021-01334-2

[4] Griffith D.T., Ashwill T.D., et al., Large offshore rotor development: design and analysis of the Sandia 100-meter wind turbine blade, 53rd Structures, Structural Dynamics and Materials Conference, 23-26 April, 2012. https://doi.org/10.2514/6.2012-1499

[5] Migliaccio G., Analytical determination of the influence of geometric and material design parameters on the stress and strain fields in non-prismatic components of wind turbines, J. Phys. Conf. Ser., 2265, 032033, 2022. https://doi.org/10.1088/1742-6596/2265/3/032033

[6] Love A.E.H., A treatise on the mathematical theory of elasticity, 4th ed., Dover Publications, 1944.

[7] Muskhelishvili N.I., Some basic problems of the mathematical theory of elasticity, Springer Verlag, 1977. https://doi.org/10.1007/978-94-017-3034-1

[8] Jourawski D.I., Sur la résistance d’un corps prismatique et d’une piéce composée en bois ou on tôle de fer à une force perpendiculaire à leur longeur. Annales Des Ponts Et Chaussées, 12, 328-351, 1856.

[9] Paglietti A., Carta G., Remarks on the current theory of shear strength of variable depth beams, Open J. Civ. Eng., 3, 28-33, 2009. https://doi.org/10.2174/1874149500903010028

[10] Balduzzi G., Hochreiner G., Füssl J., Stress recovery from one dimensional models for tapered bi-symmetric thin-walled I beams: deficiencies in modern engineering tools and procedures, Thin-Walled Struct., 119, 934-945, 2017. https://doi.org/10.1016/j.tws.2017.06.031

[11] Bleich F., Stahlhochbauten Bd. 1, Springer, Berlin, 1932. https://doi.org/10.1007/978-3-642-99613-9

[12] Pugsley A.G., Weatherhead R.A., The shear stresses in tapered beams, Aircr. Eng., 46, 218-226, 1942. https://doi.org/10.1017/S0368393100099090

[13] Krahula J.L., Shear formula for beams of variable cross section, AIAA Journal, 13(10), 1390-1391, 1975. https://doi.org/10.2514/3.6996

[14] Bertolini P, Eder M.A., et al., Stresses in constant tapered beams with thin-walled rectangular and circular cross sections, Thin-Walled Struct., 137, 527-540, 2019. https://doi.org/10.1016/j.tws.2019.01.008

[15] Migliaccio G., Ruta G., et al., A new shear formula for tapered beamlike solids undergoing large displacements, Meccanica, 2022. https://doi.org/10.1007/s11012-022-01529-1

[16] Migliaccio G., Analytical evaluation of stresses and strains in inhomogeneous non-prismatic beams undergoing large deflections, Acta Mech., 2022. https://doi.org/10.1007/s00707-022-03247-x