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The extended membrane analogy for an engineered evaluation of the torsional properties of multi-material beams

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Abstract. Prandtl's membrane analogy for cylindrical bars under torsion is extended to beams with multi-material cross section of any geometry. It was demonstrated that the problem is governed by equations formally analogue to those describing the deformation of an inflated membrane, differently tensioned in the regions corresponding to the cross-section domains hosting different materials. The analogy allows to evaluate not only the state of stress in the composite bar, but also its torsional stiffness. Here, the proposed method is used to evaluate the torsional response of a wide range of geometries, corresponding to glued multi-material bars, layered composites and laminated glass with reinforcements.

Introduction

There is a renewed and growing interest for non-homogeneous, multi-material, sandwich and laminated beams and bars. It is important to assess their response to torsion because they can be used as shafts, or at risk of flexural-torsional buckling. Elements of this type are usually verified using three-dimensional Finite Element (FE) models, requiring a consistent computational effort and a particular attention at the interface and boundary conditions at the end sections.

It is well known that the linear elastic problem à la De Saint Venant for a cylindrical bar under torsion is governed by differential equations and boundary conditions presenting a formal analogy with those describing the small deformation of an inflated initially-flat membrane stretched by an uniform equibiaxial stress, shaped as the cross section of the bar and constrained at the border. This *membrane* or *soap-film analogy*, ingeniously first proposed by Prandtl in 1903 [1], has been recently extended [2] to multi-material cross sections of any geometry. Specifically, it has been demonstrated that the problem is governed by the same equations describing the deformation of an inflated membrane, differently tensioned in the regions corresponding to the cross-section domains hosting different materials, under a stress state inversely proportional to shear moduli of the materials. The torsional stiffness is related to the volume enclosed between the deformed membrane and its reference flat configuration. The model allows to recover the case of multi-connected cross sections, by considering the holes as material domains with vanishing elastic stiffness. In the analogy, these correspond to infinitely taut regions of the membrane, that remain plane when pressurized.

Here, we present the practical use of the extended membrane analogy, to estimate the torsional stiffness in three different kind of composites: multi-material bars with circular symmetry, layered composites and laminated glass strengthened by stiff elements. For the simplest case, the multi-tensioned membrane problem can be solved in closed from, while for more complex geometries the membrane problem is implemented in a commercial FE code. In general, the numerical model of the membrane presents no difficulty, and it is certainly much easier than implementing a three-dimensional FE model for the whole bar.

The membrane analogy

Consider the De Saint–Venant's problem for a prismatic bar, whose cross section is composed by N different materials, with shear modulus G_i (i=1,…, N), connected by smooth interfaces, subjected to a load distribution at the ends equipollent to a torque T. With reference to Fig. 1, introduce the right-handed reference system (x,y,z), with z coincident with the bar axis, and denote by Ω_i the domain hosting the ith material, by Γ_{i0} the intersection of the boundary of Ω_i with the external boundary, and by Γ_{ij} the interface between materials i and j ($i \neq j$). To distinguish 'open' and 'closed' bi-material interfaces, the latter will be formally denoted by Γ_{ij}^0.

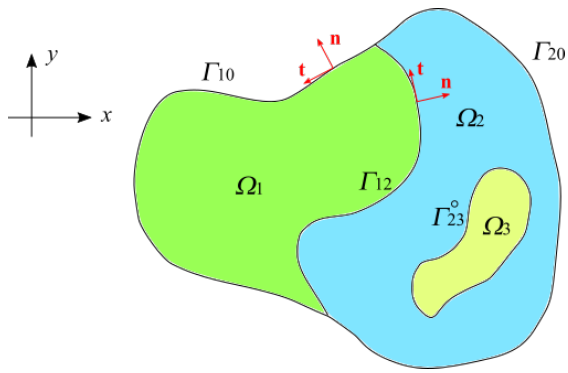


Figure 1. *Scheme of a multi-material cross section.*

In analogy with the classical approach by Prandtl, introduce, for each material domain, the stress function $F_i(x,y)$, such that the shear stresses take the form

$$\tau_{xz,i} = \frac{\partial F_i(x,y)}{\partial y} \quad \text{and} \quad \tau_{yz,i} = \frac{\partial F_i(x,y)}{\partial x}. \tag{1}$$

Using the same arguments of the classical approach, for the ith material,

$$\Delta F_i(x,y) = -2G_i\Theta \;\; in \;\; \Omega_i \quad \text{and} \quad F_i(x,y) = 0 \;\; on \;\; \Gamma_{i0}, \tag{2}$$

Where $\Delta(\cdot) = \partial^2(\cdot)/\partial x^2 + \partial^2(\cdot)/\partial y^2$, and Θ is the torsion angle per unit length. Bi-material matching conditions [3] require the continuity of the normal traction, and of the strain in tangential direction. In terms of stress function, this may be written as [2]

$$F_i(x,y) = F_j(x,y) \quad \text{and} \quad \frac{1}{G_i}\frac{\partial F_i(x,y)}{\partial \mathbf{n}} = \frac{1}{G_j}\frac{\partial F_j(x,y)}{\partial \mathbf{n}} \quad on \;\; \Gamma_{ij}. \tag{3}$$

where **n** denote outward normal vector on Γ_{ij}, as depicted in Figure 1. If the jth material domain Ω_j is enclosed by a closed curve Γ_{ij}^0, recalling (2), one obtains [2]

$$\frac{1}{G_j}\oint_{\Gamma_{ij}^0} \frac{\partial F_i(x,y)}{\partial \mathbf{n}}\, dl = -2A_j\Theta. \tag{4}$$

Once the stress function has been determined, the torque T and the torsional stiffness k are obtained as

$$T = 2\sum_{i=1}^{N}\int_{\Omega_i} F_i(x,y)\, dA_i, \quad k = \frac{2}{\Theta}\sum_{i=1}^{N}\int_{\Omega_i} F_i(x,y)\, dA_i. \tag{5}$$

Consider now an inflated membrane, defined by the same domain Ω, subjected to a uniform pressure *p,* whose (equibiaxial) tension is not uniform: the membrane portions corresponding to the domains Ω_i are subjected to the tension $s_i = 1/(2G_i)$. Denoting with $w_i(x, y)$ the out-of-plane displacement of the portion Ω_i, the governing equations read

$$\Delta w_i(x,y) = -p/s_i \quad in \quad \Omega_i \quad \text{and} \quad w_i(x,y) = 0 \quad on \quad \Gamma_{i0}\,, \tag{6}$$

with interface conditions

$$w_i(x,y) = w_j(x,y) \quad \text{and} \quad s_i \frac{\partial w_i(x,y)}{\partial \mathbf{n}} = s_j \frac{\partial w_j(x,y)}{\partial \mathbf{n}} \quad on \quad \Gamma_{ij}\,, \tag{7}$$

The first of (7) states the continuity of the membrane displacement at the interface, while the second may be interpreted by considering that, under the hypothesis of small out-of-plane displacements, $s_i\, \partial w_i(x,y)/\partial \mathbf{n}$ represents the out-of-plane force per unit length exerted on the interface by the tension of the membrane portion Ω_i. Therefore, it corresponds to the *out-of-plane* equilibrium condition for the interface Γ_{ij}. Observe that the *in-plane* equilibrium of the interface cannot be satisfied (except for the trivial case $s_i = s_j$). Hence, the membrane necessitates constrains in correspondence of the interface, which allow out-of-plane movements but refrain in-plane displacements, providing a constraint reaction that re-establishes in-plane equilibrium.

If the contour Γ_{ij}^0 is closed, it may be verified that

$$s_j \oint_{\Gamma_{ij}^0} \frac{\partial w_i(x,y)}{\partial \mathbf{n}}\, dl = -pA_j\,, \tag{8}$$

which corresponds to the condition of global equilibrium in the out-of-plane direction of the whole membrane portion Ω_j, subjected to a total load pA_j.

There is a formal analogy between equations (2-4), governing the torsional response of the multi-material bar, and the "multi-tensioned" membrane equations (6-8). As in the classical Prandtl's analogy, the torque *T* and the torsional stiffeness *k* (5) correspond to the double of the volume *V* contained between the deformed membrane and its projection on the (x, y) plane, and to $2V/p$, respectively. As discussed in [2], the analogy may be extended to multiply connected cross sections, by considering the holes as domains hosting a material with vanishing elastic modulus, corresponding in the analogy to regions where the membrane is infinitely taut, i.e., $s_j \to \infty$.

Practical application

For simplest geometries, the problem can be solved in closed form, providing exact results. For most complicated cases, the analogy is very useful, but not for an experimental verification, as originally conceived by Prandtl, which is possible but not easy. More conveniently, the membrane problem can be directly implemented in a commercial FE code, allowing to consider membrane portions with different tautness. Therefore, one can determine the torsional stiffness for multi-material sections of any kind with a two-dimensional numerical model, at the same time evaluating the stress state, and use the result in a subsequent structural model based on 1D beam elements. This procedure simplifies the common engineering approach based on a 3D FE model, requiring a consistent effort for meshing and a particular attention at the interface and boundary conditions at the beam ends.

Working Examples

Example 1: Glued multi-material bar with circular symmetry.

Consider the multi-material bar whose cross section is shown in Fig. 2a, composed of an inner cylindrical bar of radius R_{23} and an external hollow cylinder with internal radius R_{12} and external radius R_{01}, both made of steel (G_1=G_3= 80 GPa), glued by a thin layer of glue, of thickness *t* and mean radius R_m, with variable shear modulus G_2. Thanks to the circular symmetry, both the torsional and the membrane problems, associated with eq,s (2-3) and (6-7), respectively, can be solved in closed form. The volume enclosed in the deformed membrane is given by

$$V = \frac{\pi p}{8}\left[\frac{\left(R_{01}^4 - R_{12}^4\right)}{s_1} + \frac{\left(R_{12}^4 - R_{23}^4\right)}{s_2} + \frac{R_{23}^4}{s_3}\right], \tag{9}$$

Hence, the torsional stiffness (5) is

$$k = \frac{2V}{p} = \frac{\pi}{4}\left[\frac{\left(R_{01}^4 - R_{12}^4\right)}{s_1} + \frac{\left(R_{12}^4 - R_{23}^4\right)}{s_2} + \frac{R_{23}^4}{s_3}\right] = \frac{\pi}{2}\left[G_1\left(R_{01}^4 - R_{12}^4\right) + G_2\left(R_{12}^4 - R_{23}^4\right) + G_3 R_{23}^4\right]. \tag{10}$$

Fig. 2b shows k as a function of G_2, for R_{01}= 20 mm and R_m=10 mm (obviously, R_{23}= R_m-t/2 and R_{12}= R_m+t/2), for different values of t, together with the qualitative deformed shape of the corresponding membranes, for different values of G_2. It is evident that the influence of the glue stiffness on the overall torsional stiffness is quite limited.

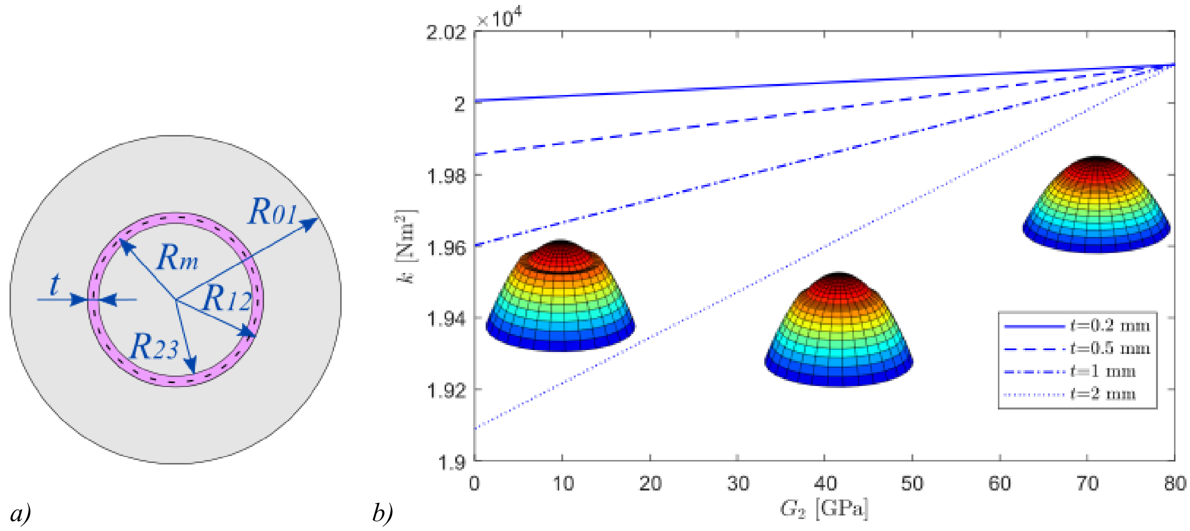


Figure 2. *Example 1. a) Considered geometry, and b) torsional stiffness as a function of G_2, with qualitative membrane deformed shape for $G_2=G_1/100$, $G_2=G_1/2$, and $G_2=G_1$.*

<u>*Example 2: triple laminated glass element.*</u>

Consider now the triple laminated glass element of Fig. 3a, with b=200 mm, composed of three glass plies (h =12 mm, G_{glass} =28.69 GPa), bonded by two interlayers of thickness t = 1.52 mm. To evaluate the influence of the interlayer properties on the torsional response, its shear modulus G_{int} is made to vary between 0 (correspondent to infinitely taught membrane) and G_{glass}.

To numerically evaluate the torsional stiffness, a 3D model has been implemented with the software ABAQUS. A relative rigid rotation of 0.1 rad has been prescribed to the extreme cross sections of a 1m long beam, while allowing the warping of the cross section. The body has been discretized with quadratic solid elements with reduced integration. The membrane model has been implemented by using linear quadrilateral membrane elements with reduced integration, with the same mesh used for the beam cross section. Null displacements have been prescribed on the external boundary, while at the inner interfaces only the in-plane displacements are prevented.

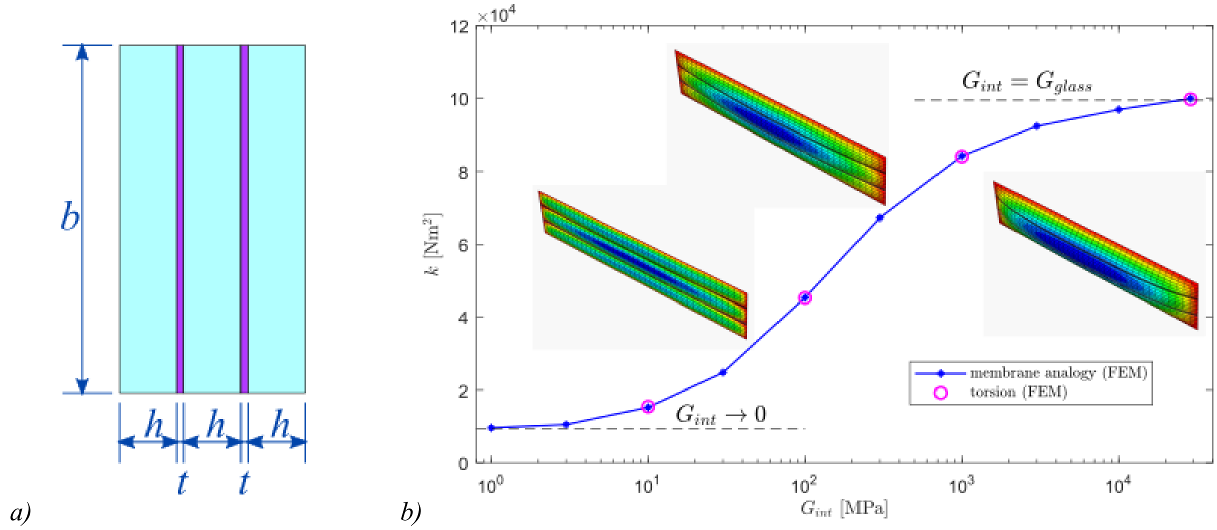


Figure 3. *Example 2. a) Considered geometry, and b) torsional stiffness as a function of G_{int}, with qualitative membrane deformed shape for G_{int}=10 MPa, G_{int}=100 MPa, and G_{int}=1000 MPa.*

Fig. 3b shows the comparison of the torsional stiffness evaluated by means of the torsion FE analyses (circles), and with the membrane analogy (stars). The results are in very good agreement (max error of the order of 1%). The analytically evaluated bounding values, corresponding to the cases of interlayer with vanishing stiffness ($G_{int} \to 0$) and of a full-glass section ($G_{int} = G_{glass}$) are plotted on the same graph. Remarkably, the computational effort required to numerically solve the membrane problem is of about 1% of that required to solve the 3D torsion problem.

<u>Example 3</u>: triple laminated glass element, with steel reinforcement.
Laminated glass beams and fins are often reinforced by steel elements, to avoid brittle response and to enhanced their failure behavior [4]. Here, the same triple laminated glass element studied in Example 2, reinforced by a steel element of width w=10 mm, as shown in Fig. 4, is considered.

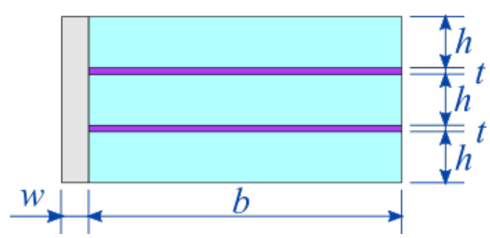
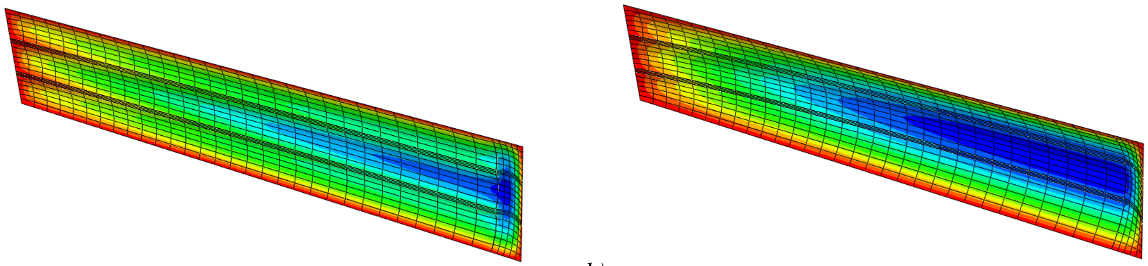


Figure 4. *Example 3. Considered geometry.*

The numerical evaluation of the torsional stiffness has been conducted as described for Example 2, by considering two different values for the interlayer stiffness, i.e., G_{int}=10 MPa, and G_{int}=100 MPa. The qualitative membrane deformed shape for these cases are shown in Fig. 5.



a) *b)*

Figure 5. *Example 3Membrane deformed shape for a) G_{int}=10 MPa, b) G_{int}=100 MPa.*

Table 1 records the torsional stiffness *k*, evaluated either with FE analysis of the torsion problem and with the membrane analogy. Again, the results are in very good agreement. The computational effort required to numerically solve the membrane problem is of about 0.6% of that required to solve the 3D torsion problem.

Table 1. *Example 3. Torsional stiffness for G_{int}=10 MPa and G_{int}=100 MPa.*

	k (FEM-torsion) [Nm^2]	*k* (FEM-membrane) [Nm^2]	Error [%]
G_{int}= 10 MPa	31615.95	31604.00	0.038
G_{int}= 100 MPa	76467.32	76558.25	-0.119

Conclusions

The semi-analytical method based on an extension of the classical Prandtl's membrane analogy, allows to readily evaluate the torsional properties of multi-material bars. This method replaces the 3D FE analysis of the torsion problem for the bar, which is time-consuming due to the complex shape and the high numbers of materials, with a FE model of the equivalent membrane structure. In the worked examples, this required about the 1% of the computational effort for the 3D case.

The practical use of this method has been here demonstrated for a wide range of geometries, corresponding to glued multi-material bars, layered composites and laminated glass beams with reinforcements. The worked examples confirm the potential of this approach in providing both a qualitative (intuitive) and quantitative indication of the torsional properties, and its advantages in the practice of structural design. We believe that the method here proposed may represent a powerful tool to engineers for the characterization of the torsional response of composite and layered structures, also in view of their verification against buckling under bending.

References

[1] Prandtl L., Zur torsion von prismatischen stäben. Physikalische Zeitschrift 4 (1903) 758–770.

[2] Galuppi, L., Royer-Carfagni, G., Membrane analogy for multi-material bars under torsion, Proc. R. Soc. A 475 (2019) 20190124. https://doi.org/10.1098/rspa.2019.0124

[3] Dundurs J, Sotiropoulos D. On stress conditions and computations at bi-material interfaces. Comput. Mech. 21 (1998) 300–305. https://doi.org/10.1007/s004660050306

[4] Martens, K., Caspeele, R., Belis, J. Development of reinforced and posttensioned glass beams: review of experimental research. J. Struct. Eng. 142.5 (2016) 04015173. https://doi.org/10.1061/(ASCE)ST.1943-541X.0001453