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On the Coriolis Effect for Internal Ocean Waves

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Abstract. A derivation of the Ostrovsky equation for internal waves with methods of the Hamiltonian water wave dynamics is presented. The internal wave formed at a pycnocline or thermocline in the ocean is influenced by the Coriolis force of the Earth's rotation. The Ostrovsky equation arises in the long waves and small amplitude approximation and for certain geophysical scales of the physical variables.

Introduction

The internal ocean waves could have a significant impact on offshore engineering structures, such as oil platforms in the oceans as well as stationary tubes for oil and gas transportation stretching along the ocean shelf slope [1]. Builders of underwater constructions in equatorial districts, for example, experience the influence of huge underwater internal waves and strong surface flows, which are interfering with their work activities.

The internal waves often are driven by tidal motion. The open water exploitation of tidal and wave power is under current considerations. It has been estimated globally that 180 TWh of economically accessible tidal energy is available. However, due to geographical, technical, and environmental constraints, only a fraction of this could be captured in practical terms [5].

The pattern of the ocean movement around the points of zero tidal wave amplitude (amphidromic point) is due to the Coriolis effect. Therefore there are deep interrelations between the tidal motion, internal waves and Coriolis forces that deserve detailed studies, since these are of potential practical significance.

In this work we examine the Coriolis effect on the internal wave propagation following the idea of *nearly* Hamiltonian approach, developed in series of previous papers like [6, 4] and [3] and generalising the Hamiltonian approach of Zakharov [14].

A mass of moving air or water subject only to the Coriolis force travels in a circular trajectory called an *inertial circle*, for the atmosphere see the illustration on Fig. 1(a). For ocean waves the Coriolis Effect is not so pronounced, nevertheless it affects the wave propagation. For ocean waves of large magnitude, the viscosity does not play an essential role and can be neglected, so effectively the fluid dynamics is govern by Euler's equation.

Internal Waves with Coriolis force - the Setup

The Euler equation with included Coriolis force is

$$V_t + (V \cdot \nabla)V + 2\vec{\omega} \times V = -\frac{1}{\rho}\nabla p + \vec{g} \tag{1}$$

where the velocity vector field $V = (u, v, w)$ is presented through its components in a local coordinate system where the geophysical axis x is oriented to the East, the y axis is pointing to the North and the z axis is vertical to the Earth's surface, $\vec{g} = (0,0,-g)$ is the Earth's gravity acceleration. In addition we have incompressibility, i.e. div $V = 0$. p is the pressure in the fluid. The Earth's angular velocity at latitude in this system is $\vec{\omega} = \omega\,(0, \cos\theta, \sin\theta)$, $\omega = 7.3 \times 10^{-5}$ rad/s. Introducing the parameters $f = 2\omega \sin\theta$ and $r = 2\omega\cos\theta$ we have

$$2\vec{\omega} \times V = (rw - fv, fu, -ru)\ .$$

For Equatorial motion $\theta = 0$ and $f = 0$ so there are no forces acting in the y-direction. Moreover, the Coriolis forces are supporting the fluid to move along the Equator (in the x-direction), so that its motion remains two-dimensional. Such situation with internal equatorial waves and currents is studied in [4].

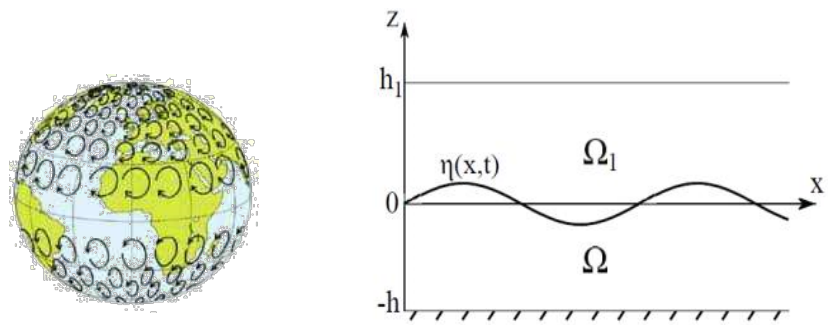

Fig. 1: (a) Left: Coriolis forces in the atmosphere. Schematic representation of inertial circles of air masses in the absence of other forces. Source: Wikipedia; (b) Right: System with an internal wave. The fluid domain Ω contains fluid of higher density. The pycnocline/thermocline separates the two fluid domains Ω and Ω 1. The function η(x, t) describes the elevation of the internal wave.

We are going to consider now for example θ > 0. In addition we assume that the fluid motion is irrotational (i.e. absence of currents and vorticity), apart from the global rotation caused by the Coriolis forces. In this approximation the velocity field is potential, i.e. $\mathbf{V} = \boldsymbol{\nabla\varphi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$ and the Coriolis Effect will be presented as a perturbation to the potential motion. The governing equations (1) acquire the form:

$$\left(\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz\right)_x + r\varphi_z - f\varphi_y = 0,$$

$$\left(\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz\right)_y + f\varphi_x = 0, \qquad (2)$$

$$\left(\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz\right)_z - r\varphi_x = 0,$$

where p is the pressure in the fluid. The internal waves are illustrated on Fig. 1(b). For fixed y the system is bounded at the bottom by an impermeable flatbed and is considered as being bounded on the surface by an assumption of absence of surface motion. The domains Ω and Ω_1 are defined with values associated with each domain using corresponding respective subscript notation. Also, subscript c (implying common interface) will be used to denote evaluation on the internal wave $z = \eta(x, t)$. Propagation of the internal wave is assumed to be in the positive x-direction which is considered to be eastward. The function $\eta(x, t)$ describes the elevation of the internal wave with the spatial mean of $\eta(x,t)$ assumed to be zero. The system is considered incompressible with ρ and ρ_1 being the respective constant densities of the lower and upper media and stability is given by the immiscibility condition $\rho > \rho_1$. For long internal waves the parameter $\delta = \frac{h}{\lambda} \ll 1$ and φ is a small quantity of order δ, see for example [4]. The terms proportional to r lead to a very small correction (0.01%) of the wave propagation speed c_0 in the x-direction, for the feasible values of the parameters (see for example the calculations in [4]) and thus they can be neglected. The motion in the y direction is very slow in comparison to the wave propagation in the x-direction, therefore in leading order we have $p = p(x, z)$ and we can use the second equation in Eq. (2) in linear approximation to exclude the y dependence, $\varphi_{ty} + f\varphi_x = 0$ giving formally

$$\varphi_y = -f\partial_t^{-1}\varphi_x.$$

Assuming further that f is of order $\delta^2 \ll 1$ and noting that the ∂_x operator with an eigenvalue $k = \frac{2\pi}{\lambda}$ is also of order δ, for compatible time-scales $\partial_t \sim \delta$ thus we see that the y-derivative of φ is $\varphi_y \sim \delta\varphi_x$ (more details about the scales could be found in [4]). The first equation in (2) gives the following generalisation of the Bernoulli equation:

$$\varphi_t + \frac{|\nabla\varphi|^2}{2} + \frac{p}{\rho} + gz + f^2\partial_t^{-1}\varphi = 0.$$

Therefore in the nonlinear contribution $|\nabla\varphi|^2 = \varphi_x^2 + \varphi_y^2 + \varphi_z^2$ the first term is $\sim\delta^4$ already small and the second is $\sim\delta^6$ (much smaller) and could be neglected in comparison to φ_x^2 giving $|\nabla\varphi|^2 \approx \varphi_x^2 + \varphi_z^2$ or

$$\varphi_t + \frac{\varphi_x^2+\varphi_z^2}{2} + \frac{p(x,z)}{\rho} + gz + f^2\partial_t^{-1}\varphi = 0. \tag{3}$$

We can proceed now with this effectively $(2+1)$-dimensional equation for the x and z dependent variables, considering y fixed, since there are no y-derivatives.

(Nearly) Hamiltonian representation of the internal wave dynamics

The propagation of the internal wave is assumed to be in the positive x-direction which is considered to be *eastward*. At $z = \eta(x, t)$ we have $p(x, \eta, t) = p_1(x, \eta, t)$ and therefore Eq. (3) gives the Bernoulli condition

$$\rho\left((\varphi_t)_c + \frac{(\varphi_x^2+\varphi_z^2)_c}{2} + g\eta + f^2(\partial_t^{-1}\varphi)_c\right) = \rho_1\left((\varphi_{1,t})_c + \frac{(\varphi_{1,x}^2+\varphi_{1,z}^2)_c}{2} + g\eta + f^2(\partial_t^{-1}\varphi_1)_c\right)$$

The last equation suggests the introduction of the variable $\xi(x,t) = (\rho\varphi - \rho_1\varphi_1)_c$. Indeed, following [6, 4] this equation can be written in the nearly Hamiltonian form

$$\xi_t = -\frac{\delta H_0}{\delta\eta} - (\delta^2) f^2(\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1))_c \tag{4}$$

for the Hamiltonian (expansion with respect to the scale parameter δ, following the leading order; $D = -i\partial_x \sim \delta, \eta \sim \delta^2$)

$$H_0(\xi,\eta) = \frac{1}{2}\int_{\mathbb{R}} \xi D(\alpha_1 + \delta^2(\alpha_3\eta - \alpha_2 D^2))D\xi dx + \frac{1}{2} g(\rho - \rho_1)\int_{\mathbb{R}} \eta^2 dx \tag{5}$$

where

$$\alpha_1 = \frac{hh_1}{\rho_1 h + \rho h_1},\ \alpha_2 = \frac{h^2 h_1^2(\rho h + \rho_1 h_1)}{3(\rho_1 h + \rho h_1)^2}, \qquad \alpha_3 = \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^2}. \tag{6}$$

The kinematic boundary condition on the interface leads to the second equation,

$$\eta_t = \frac{\delta H_0}{\delta\xi} \tag{7}$$

so that Eq. (4) and Eq. (7) represent the *nearly* Hamiltonian formulation of the internal wave dynamics in the long-wave -small amplitude approximation.

Boussinesq and KdV type approximations. Ostrovsky equation

Introducing the variable $\tilde{u} = \xi_x$ one can verify by a simple computation that $\partial_x(\partial_t^{-1}(\rho\varphi - \rho_1\varphi_1))_c = \partial_t^{-1}\tilde{u} +$ smaller order terms. Then the equations (4) and (7) in terms of η and $\tilde{u} = \xi_x$ are

$$\eta_t + \alpha_1\tilde{u}_x + \delta^2\alpha_2\tilde{u}_{xxx} + \delta^2\alpha_3(\eta\tilde{u})_x = 0, \tag{8}$$

$$\tilde{u}_t + g(\rho - \rho_1)\eta_x + \delta^2\alpha_3\tilde{u}\tilde{u}_x + \delta^2 f^2(\partial_t^{-1}\tilde{u}) = 0. \tag{9}$$

In leading order $\eta_t + \alpha_1\tilde{u}_x = 0$, $\tilde{u}_t + g(\rho - \rho_1)\eta_x = 0$ or $\eta_{tt} = -\alpha_1\tilde{u}_{xt} = g\alpha_1(\rho - \rho_1)\eta_{xx}$, $\eta_{tt} - g\alpha_1(\rho - \rho_1)\eta_{xx} = 0$, which is the wave equation for η giving the wave speed

$$c_0 = \pm\sqrt{g\alpha_1(\rho - \rho_1)}.$$

For an observer, moving with the flow, i.e. there are left- (minus sign) and right-running (+ sign) waves. Moreover, in the leading approximation, for linear waves, the functions depend on the characteristic variable $x - c_0 t$, therefore $\tilde{u} = \frac{c_0}{\alpha_1}\eta$. In the next order approximations with respect to the scale parameter δ obviously

$$\tilde{u} = \frac{c_0}{\alpha_1}\eta + \delta^2(\cdots). \tag{10}$$

however we will not need this explicitly, see for example [10]. Differentiating Eq. (8) with respect to *t*

$$\eta_{tt} + \alpha_1 \tilde{u}_{tx} + \delta^2 \alpha_2 \tilde{u}_{txxx} + \delta^2 \alpha_3 (\eta_t \tilde{u} + \eta \tilde{u}_t)_x = 0,$$

and substituting in it $\tilde{u}_t$ from Eq. (9), $\tilde{u}$ from Eq. (10) and $\eta_t = -\alpha_1 \tilde{u}_x + \delta^2(\dots)$ where necessary, neglecting δ^4 terms, we obtain the following generalised Boussinesq equation for η:

$$\eta_{tt} - c_0^2 \eta_{xx} - \delta^2 \frac{3\alpha_3 c_0^2}{2\alpha_1} (\eta^2)_{xx} - \delta^2 \frac{\alpha_2 c_0^2}{\alpha_1} \eta_{xxxx} + \delta^2 f^2 \eta = 0. \tag{11}$$

The dispersion law of this equation is $\tilde{\omega}^2(k) = c_0^2 k^2 - \delta^2 \frac{\alpha_2 c_0^2}{\alpha_1} k^4 + \delta^2 f^2$ or approximately

$$\tilde{\omega}(k) = c_0 k - \delta^2 \frac{\alpha_2 c_0^2}{2\alpha_1} k^3 + \delta^2 \frac{f^2}{2kc_0}. \tag{12}$$

Furthermore, a generalised KdV type equation of the form

$$\eta_t + c_0 \eta_x + \delta^2 a \eta_{xxx} + \delta^2 b (\eta^2)_x + \delta^2 n f^2 \partial_x^{-1} \eta = 0 \tag{13}$$

for some constants *a*, *b, n* (yet unknown) could be obtained from Eq. (11). Indeed, differentiating the above equation with respect to *t* we have

$$\eta_{tt} + c_0 \eta_{xt} + \delta^2 a \eta_{txxx} + \delta^2 b (\eta^2)_{xt} + \delta^2 n f^2 \partial_x^{-1} \eta_t = 0 \tag{14}$$

in which we substitute η_t from Eq. (13) to obtain (neglecting δ^4 terms)

$$\eta_{tt} - c_0^2 \eta_{xx} - \delta^2 2bc_0 (\eta^2)_{xx} - \delta^2 a c_0 \eta_{xxxx} - \delta^2 2nc_0 f^2 \eta = 0.$$

The comparison with Eq. (11) gives

$$a = \frac{\alpha_2 c_0}{2\alpha_1}, \qquad b = \frac{3\alpha_3 c_0}{4\alpha_1}, \qquad n = -\frac{1}{2c_0}.$$

Then finally the KdV-type equation acquires the form

$$\eta_t + c_0 \eta_x + \delta^2 \frac{\alpha_2 c_0}{2\alpha_1} \eta_{xxx} + \delta^2 \frac{3\alpha_3 c_0}{4\alpha_1} (\eta^2)_x = \delta^2 \frac{f^2}{2c_0} \partial_x^{-1} \eta.$$

This is also known as Ostrovsky equation [12]. Note that the dispersion law of the Ostrovsky equation is like in Eq. (12).

For surface waves the derivation is analogous, only the Hamiltonian H_0 is the KdV Hamiltonian for surface waves. The derivation directly from Euler's equations could be found in Leonov's paper [11]. The Ostrovsky equation itself is Hamiltonian and possesses three conservation laws, however it is not bi-Hamiltonian and it is not integrable by the Inverse Scattering Method [2]. Solutions from perturbations of the KdV solitons can be derived in principle, although this is technically difficult, see for example [8] and the references therein. Various other aspects of the equation have been studied extensively by now in numerous works, see for example [9, 13] and the references therein.

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